Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm

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Abstract

This paper studies the properties of the solution to the heterogeneous agents model in Den Haan, Judd and Juillard (2008). To solve for the individual policy rules, we use an Euler-equation method iterating on a grid of prespecified points. To compute the aggregate law of motion, we use the stochastic-simulation approach of Krusell and Smith (1998). We also compare the stochastic- and non-stochastic-simulation versions of the Krusell-Smith algorithm, and we find that the two versions are similar in terms of their speed and accuracy.

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1 Introduction

This paper studies the properties of the solution to the incomplete markets model with aggregate uncertainty in Den Haan, Judd and Juillard (2008). Our solution method consists of two interconnected steps: the first is to solve the individual problem for a given aggregate behavior of the economy and the second is to compute the aggregate law of motion for the given individual policy rules. We iterate on these two steps until we find a fixed point at which the individual and aggregate policy rules are mutually consistent.

Step one is straightforward: the individual problem is the typical capital-accumulation problem with an occasionally binding borrowing constraint, and it can be solved by the standard numerical methods. We solve the individual problem by using a grid-based Euler-equation algorithm similar to that in Maliar and Maliar (2005, 2006). We extend Maliar and Maliar's (2005, 2006) algorithm by incorporating a simple polynomial rule for constructing the grid, which allows us to vary the concentration of capital grid points on different parts of the domain, thus increasing the accuracy of approximation on non-linear parts of the policy rules. Our algorithm is also similar to the grid-based Euler-equation method used by Baxter, Crucini and Rouwenhorst (1990) for solving the standard one-sector growth model. Furthermore, our algorithm is related to the parameterized expectations algorithm used in Den Haan and Marcet (1990), Den Haan (1997), Christiano and Fisher (2000), Maliar and Maliar (2003b), and Algan, Allais and Den Haan (2008). However, the above papers parameterize an expectation term in the Euler equation and use a polynomial approximation, whereas we parameterize a capital function and compute a solution on a grid of pre-specified points.[1]

Step two is non-trivial. Decisions of each heterogeneous agent depend on the interest rate and wage rate, which in turn depend on the aggregate capital stock. Since the aggregate capital stock is determined by capital holdings of all heterogeneous agents, the whole capital distribution becomes a state variable.[2] With a continuum of agents, this distribution is a function, and therefore, it cannot be used as an argument of the individual policy rules. To

[1] For a general discussion of the Euler-equation methods, see Judd (1998).

[2] Under the assumption of complete markets, the aggregate behavior of a similar heterogeneous-agent economy with idiosyncratic and aggregate uncertainty can be described by a one-consumer model; see Maliar and Maliar (2003a) for this aggregation result. In this special case, the state space does not include the whole capital distribution but only its mean.

deal with this problem, Krusell and Smith (1998) propose to summarize the capital distribution by a discrete and finite set of moments.[3] They solve the individual problem by using value iteration, and they compute the aggregate law of motion by simulating a panel for a large finite number of agents and by running regressions on the simulated data. In this paper, we follow the stochastic-simulation approach of Krusell and Smith (1998). Consequently, our solution procedure is a variant of the Krusell-Smith algorithm, specifically one in which the individual problem is solved by an Euler-equation method instead of Krusell and Smith's (1998) value function iteration. Our computer programs are written in MATLAB in an instructive manner and are provided on the JEDC web site (see the web pages of the authors for updated versions of the program).

An important advantage of the stochastic-simulation Krusell-Smith algorithm is that it is simple, intuitive and easy to program. As Algan, Allais and Den Haan (2008) show, however, stochastic-simulation methods have two potential shortcomings. First, the introduction of stochastic simulations produces sampling noise, which makes the policy rules to depend on a specific random draw. Second, the simulated endogenous data are clustered around the mean, which implies that the accuracy of the approximation on the tails is low. They argue that replacing a stochastic simulation with a non-stochastic one can enhance the accuracy and speed of the algorithm. Therefore, it is of interest to assess the accuracy of the stochastic-simulation version of the Krusell-Smith algorithm and to compare it with a non-stochastic-simulation version.

We find that, despite the above shortcomings, the stochastic-simulation Krusell-Smith method produces sufficiently accurate solutions.[4] This is true even under our relatively small panel of $10,000$ agents and relatively short simulation length of $1,100$ periods. For example, in an accuracy test where the model was simulated on a random realization of shocks of $10,000$ periods, the average and maximum errors in our aggregate capital series were

[3] Den Haan (1997) proposes an alternative approach for dealing with this problem, namely, to parameterize the cross-sectional distribution with a polynomial.

[4] An exception is very large errors produced by our method in a dynamic Euler-equation accuracy test, see Table 14 in Den Haan (2008). A typo in our program is responsible for these large errors. After we corrected the typo, the errors became considerably lower, namely, in Table 14, the capital (scaled) average and maximum errors should be equal to 0.0319% and 0.0926%, respectively, and the consumption average and maximum errors should be equal to 0.0091% and 0.4360%, respectively.

0.050% and 0.156%, respectively. Furthermore, we consider a non-stochastic-simulation Krusell-Smith algorithm where simulations are performed on a grid of pre-specified points, as is described in the appendix in Den Haan (2008).[5] We find that the benchmark stochastic-simulation version of the Krusell-Smith algorithm with a panel of $10,000$ agents has approximately the same cost as the non-stochastic-simulation version with a grid of $1,000$ points and produces solutions of comparable (or even higher) accuracy. Thus, in our case, the introduction of non-stochastic simulation does not lead to substantial improvements.

2 The individual problem

In this section, we describe an Euler-equation algorithm for finding a solution to the individual problem described in Den Haan, Judd and Juillard (2008). This is the standard capital-accumulation problem with an occasionally binding borrowing constraint. The Euler equation, the budget constraint, the borrowing constraint and the Kuhn-Tucker conditions, respectively, are

$$c^{-\gamma} - h = \beta E\left\{ \left(c'\right)^{-\gamma}\left(1-\delta+r'\right)\right\}, \tag{1}$$

$$k' = \left(1-\tau\right) w\bar{l}\varepsilon + \mu w\left(1-\varepsilon\right) + \left(1-\delta+r\right)k - c, \tag{2}$$

$$k' \geq 0, \tag{3}$$

$$h \geq 0, \qquad hk' = 0, \tag{4}$$

where variables without and with primes refer to the current and future periods, respectively (we omit the individual superscripts for the sake of notational convenience). Here, c is consumption; k is capital; ε is an idiosyncratic shock that determines an employment status, with $\varepsilon = 1$ and $\varepsilon = 0$ representing the employed and unemployed states, respectively; h is the Lagrange multiplier associated with the borrowing constraint (3); r, w, μ and τ are the interest rate, wage rate, unemployment-benefit rate and labor-income tax

[5] This non-stochastic simulation procedure is close to the one considered in Rios-Rull (1997). A different non-stochastic-simulation procedure is proposed by Young (2008), who was the first to combine the Krusell-Smith algorithm with non-stochastic simulation. Algan, Allais and Den Haan (2008) perform a comparison of Rios-Rull's (1997), Young's (2008) and their own procedures.

rate, respectively; $\beta \in (0,1)$ is the discount factor; $\delta \in (0,1]$ is the depreciation rate of capital; $\gamma > 0$ is the utility-function parameter; and $\bar{l}$ is the time endowment.

The interest rate, wage and labor-income tax rate are given by

$$r = \alpha a \left(\frac{K}{\bar{l}L}\right)^{\alpha-1}, \qquad w = (1-\alpha)\, a \left(\frac{K}{\bar{l}L}\right)^{\alpha}, \qquad \tau = \frac{\mu u}{\bar{l}L},$$

where a is an aggregate productivity shock, which can take two values $1-\Delta^a$ and $1+\Delta^a$; $u = u(a)$ is the unemployment rate, which takes two values depending on the aggregate productivity shock, $u(1-\Delta^a)$ and $u(1+\Delta^a)$; K and $L \equiv 1-u$ are the aggregate capital and labor, respectively; and $\alpha \in (0,1)$ is the share of capital in production.

Our objective is to compute the individual policy rule for choosing the next-period capital stock k'. We restrict attention to a first-order recursive Markov equilibrium for which the individual policy rules are time-invariant functions of a current state. In an economy without aggregate uncertainty, $\Delta^a = 0$, the individual state variables are k and ε, and the individual policy rule for capital is $k' = k'(k, \varepsilon)$. This economy is first considered in Huggett (1993) and Aiyagari (1994) and can be studied using standard dynamic programming methods. In an economy with aggregate uncertainty, the state space also includes the aggregate productivity shock a and the capital holdings of all heterogeneous agents. With a continuum of agents, the distribution of capital is a function, and therefore, it cannot be used as an argument of the individual policy rule. Following Krusell and Smith (1998), we characterize the capital distribution by a set of moments m.[6] We must therefore find a time-invariant policy rule for the future capital $k' = k'(k, \varepsilon, m, a)$ that satisfies conditions $(1)-(4)$.

Using the budget constraint (2), we eliminate current and future con-

[6] For the given economy, Krusell and Smith (1998) show that the mean of the capital distribution contains essentially all the information, which is relevant for the individual decision making. This results is referred to in the literature as "approximate aggregation". We shall emphasize that approximate aggregation is a numerical result that needs not hold for other economies, and that in general, many moments in the state space might be needed for accurate solutions.

sumption from the Euler equation (1) to obtain

$$\begin{aligned}\widetilde{k}' &= \left(1-\frac{\mu u}{\bar{l}L}\right)w\bar{l}\varepsilon+\mu w\left(1-\varepsilon\right)+\left(1-\delta+r\right)k \\ &\quad -\left\{h+\beta E\left[\frac{1-\delta+r'}{\left(\left(1-\frac{\mu u'}{\bar{l}L'}\right)w'\bar{l}\varepsilon'+\mu w'\left(1-\varepsilon'\right)+\left(1-\delta+r'\right)k'-k'\left(k'\right)\right)^{\gamma}}\right]\right\}^{-1/\gamma},\end{aligned} \tag{5}$$

where $h \equiv h\left(k,\varepsilon,m,a\right)$, $k' \equiv k'\left(k,\varepsilon,m,a\right)$ and $k'\left(k'\right) \equiv k'\left(k'\left(k,\varepsilon,m,a\right)\right)$. We choose the relevant intervals for $k \in [0,k_{\max}]$ and $m \in [m_{\min},m_{\max}]$, and we discretize these intervals to construct a grid of points for $\left(k,\varepsilon,m,a\right)$. We subsequently solve equation (5) on the grid using the following iterative procedure.

- *Step I*. Fix some initial capital function, $k'\left(k,\varepsilon,m,a\right)$, on the grid. We set the initial capital function at $k'\left(k,\varepsilon,m,a\right)=0.9k$ for all k,ε,m,a.

- *Step II*. For each grid point $\left(k,\varepsilon,m,a\right)$, substitute the assumed capital function $k'\left(k,\varepsilon,m,a\right)$ in the right-hand side of (5), set the Lagrange multiplier equal to zero, $h\left(k,\varepsilon,m,a\right)=0$, and compute the new capital function, $\tilde{k}'\left(k,\varepsilon,m,a\right)$ in the left-hand side of (5). For each point on the grid for which $\tilde{k}'\left(k,\varepsilon,m,a\right)$ does not belong to $[0,k_{\max}]$, set $\tilde{k}'\left(k,\varepsilon,m,a\right)$ equal to the corresponding boundary value.

- *Step III*. Compute the capital function for the next iteration $\overset{\approx}{k}'\left(k,\varepsilon,m,a\right)$ using the following updating formula:

  $$\overset{\approx}{k}'\left(k,\varepsilon,m,a\right)=\eta\tilde{k}'\left(k,\varepsilon,m,a\right)+\left(1-\eta\right)k'\left(k,\varepsilon,m,a\right), \tag{6}$$

  where $\eta \in (0,1]$ is an updating parameter.

Iterate on *Steps II* and *III* until the maximum difference between $\overset{\approx}{k}'\left(k,\varepsilon,m,a\right)$ and $k'\left(k,\varepsilon,m,a\right)$ is less than a given degree of precision, which in our case was set at 10^{-8}.

We now discuss several issues related to the algorithm. By construction, the capital function $k'(k, \varepsilon, m, a)$ satisfies conditions $(1) - (3)$ and the complementary slackness condition in (4). However, we still need to check that the Lagrange multiplier $h(k, \varepsilon, m, a)$ is non-negative for each grid point (k, ε, m, a). Notice that since $\gamma > 0$, the term $\{h + \beta E[\cdot]\}^{-1/\gamma}$ in (5) is decreasing in h. Given that the unconstrained solution obtained under $h = 0$ violates the borrowing constraint and that capital on the left side of (5) must increase to satisfy the borrowing limit, we can preserve the equality sign in (5) only by increasing the Lagrange multiplier. Hence, our method guarantees that the Lagrange multiplier is always non-negative.

Regarding the upper bound $k_{\max}$, note that there is an ergodic set for k, which indicates that there exists a value $k_{\max}^{erg}$ such that the agent chooses k' inside the interval $[0, k_{\max}^{erg}]$ at all grid points. However, using $k_{\max}^{erg}$ as $k_{\max}$ leads to a grid that is too big, in the sense that the upper values of such grid have an extremely low probability of occurring in simulations. We can therefore save on computational costs by using a $k_{\max}$ that is smaller than $k_{\max}^{erg}$ but is still sufficiently large as to never be reached during simulations. In our numerical analysis, we used $k_{\max} = 1000$, and we found ex post that the simulated individual capital series never reached even the level of 500, which indicates that $k_{\max} = 1000$ is acceptable.

Furthermore, as is indicated in *Step* 2, we bound k' by $k_{\max}$ whenever it exceeds the grid, which ensures that the individual capital is always inside the interval $[0, k_{\max}]$. Alternatively, we can extrapolate the individual policy rule outside the interval $[0, k_{\max}]$. Since the latter alternative is more costly, and the properties of the policy rule in the region near $k_{\max}$ play a minor role in the solution, we adhere to the former, simpler alternative.

Concerning the number of grid points and their placement, it has been known since Huggett (1993) and Aiyagari (1994) that individual policy rules in problems with borrowing constraints have kinks near the borrowing constraints, but are close to linear at higher levels of capital. To accurately approximate the individual policy rule at low levels of capital, many grid points are thus necessary, while an accurate approximation at high levels of capital requires relatively few grid points. To take into account the above regularity, we propose the following simple polynomial rule for the placement of grid points

$$z_j = \left(\frac{j}{J}\right)^{\theta} k_{\max} \quad \text{for } j = 0, 1, ..., J, \tag{7}$$

where $J+1$ is the number of grid points with $J \geq 1$, and $\theta > 0$ is a degree of the polynomial. The rule (7) is normalized so that $z_0 = 0$ and $z_J = k_{\max}$. If $\theta = 1$, we obtain grid points that are distributed uniformly in the interval $[0, k_{\max}]$; if θ is increased, the concentration of grid points in the beginning of the interval increases while the concentration of grid points toward the end of the interval decreases.

To determine the degree of the polynomial θ that leads to the most accurate solution for a given number of grid points, we first compute an "accurate" solution by considering $100,000$ grid points uniformly placed in the interval $[0, k_{\max}]$. We then compute "approximate" solutions by considering 100 grid points, placed according to rule (7) using various values of θ. We then examine the average and maximum percentage errors between the capital choices under the "accurate" and "approximate" solutions. We find that the smallest errors are obtained under the polynomial degree $\theta = 7$: the average error was 0.0002% in this case, and the maximum error was 0.09%. We thus choose a 100-point grid with $\theta = 7$, as the benchmark. We also investigate the relationship between the solution's accuracy and the number of grid points, and we find that increasing the number of grid points from 100 to 400 augments the accuracy of the solution by about one order of magnitude.

We find that the properties of the solution can significantly depend on a specific interpolation procedure used for evaluating the decision rules off the grid. To compute the capital function off the grid, we try both a linear and cubic polynomial interpolation. In our case, the cubic polynomial interpolation is about three times slower than the linear interpolation but produces considerably more accurate solutions. Given restrictions on computational cost, we therefore face a trade-off between a linear interpolation with many points and a cubic interpolation with fewer of points. After running a number of experiments, we conclude that the cubic interpolation with fewer points is superior to the linear one with a large number of points, especially in areas where the policy rules are non-linear.

3 The stochastic-simulation algorithm

In this section, we discuss a version of the stochastic-simulation Krusell-Smith algorithm for solving the model with aggregate uncertainty. We parameterize the Aggregate Law of Motion (ALM) for a set of moments of the capital distribution, m, by the following flexible functional form

$$m' = f(m, a; b), \tag{8}$$

where b is a vector of the ALM coefficients. Subsequently, we compute b by using the following iterative procedure.

Algorithm 1 (stochastic simulation).

- *Step I.* Fix an initial vector of coefficients b. Generate and fix time series of length T for the aggregate shocks. Fix the initial distribution of capital across N heterogeneous agents. For each agent, generate and fix a time series of length T for the idiosyncratic shocks.
- *Step II.* Given b and ALM (8), compute a solution to the individual problem as described in Section 2.
- *Step III.* Use the individual policy rules computed in *Step II* to simulate the economy T periods forward by explicitly solving for the capital holdings of each agent $i = 1, ..., N$, and by calculating the set of statistics m_t for each $t = 1, ..., T$.
- *Step IV.* Regress the time series for the statistics m_{t+1} as calculated in *Step III* on the functional form $f(m_t, a_t; b)$, and call the regression coefficients $\tilde{b}$.
- *Step V.* Compute the ALM coefficients for next iteration by using updating:

  $$\overset{\approx}{b} = \lambda \tilde{b} + (1 - \lambda)\, b, \tag{9}$$

  where $\lambda \in (0, 1]$ is an updating parameter.

Iterate on *Steps* $II - V$ until the average squared difference between $\overset{\approx}{b}$ and b is less than a given degree of precision, which we set 10^{-8}.

In our experiments, we take m to be either the first moment (mean) or the first and second moments (mean and variance) of the capital distribution.

We assume that for each aggregate state, ALM (8) is a linear function of moments. For the mean, we consider a grid of four uniformly distributed values in the interval from 75% to 125% of the capital mean of the ergodic distribution, and for the variance, we consider a grid of four uniformly distributed values in the interval from 10% to 500% of the capital variance of the ergodic distribution. In fact, the above ranges of the grid values are substantially larger than those implied by the ergodic distribution. This is because the moments can deviate significantly from their ergodic values on initial iterations when the solution is inaccurate, whereas our interpolation procedure requires the moments to always be inside the grid. A more accurate -and more expensive- alternative would use narrower grids for the moments and apply extrapolation outside the grids.

In the benchmark case, we consider an economy populated by $N = 10,000$ agents, and we set the length of simulations at $T = 1,100$. In order to simulate the economy forward, we use a MATLAB interpolation routine "interpn" under the "cubic" interpolation option. The effect of initial conditions vanishes slowly over time, so that the solution to the model effectively depends on the initial assumption of capital distribution. To ensure that our initial distribution of capital comes from an ergodic set, we first solve the model by assuming a uniform distribution, then re-compute the solution using the resulting terminal distribution as a starting point. To further mitigate the effect of initial condition, we discard the first 100 periods when re-estimating ALM (8) in *Step IV*. We report the results only for the one-moment solution, because the series produced by the one- and two-moment ALM parameterizations are practically indistinguishable.

For the one-moment solution, the ALM for the bad and good aggregate states are, respectively,

$$\ln\left(K_{t+1}\right) = 0.123815 + 0.965565 \ln\left(K_t\right),$$

and

$$\ln\left(K_{t+1}\right) = 0.137800 + 0.963238 \ln\left(K_t\right).$$

Both regressions have R^2 in excess of 0.9999; however, Den Haan (2007) shows that R^2 is not a sensible measure in the context of the ALM accuracy, and that solutions with high R^2 values may still be inaccurate according to more appropriate accuracy measures.[7]

[7] Den Haan and Rendahl (2008) report that two solutions to the model, both of which

Den Haan (2007) proposes a powerful accuracy test which compares two aggregate capital series: the first is obtained by simulating a panel of agents using the individual policy rules, and the second is produced by the ALM. For a simulation of $10,000$ periods on a random realization of shocks, our stochastic-simulation method generates average and maximum error of 0.050% and 0.156%, respectively. These errors are relatively low; see Den Haan (2008, Table 16) for the results of this test for other computational methods. To illustrate the errors produced by our algorithm, we plot the first $1,000$ periods for the two simulated capital series in Figure 1, and we see that they are practically indistinguishable.

As a further accuracy check, we repeat the above test under a peculiar sequence of the aggregate productivity shock, in which the economy remains in a bad state for the first 100 periods, then shifts into in a good state for the next 100 periods. Even though this peculiar realization of shocks is very different from the one used in computing the solution, the average and maximum errors are still very low: 0.062% and 0.146%, respectively. For this experiment, the aggregate capital series generated by the individual policy rule and by the ALM are shown in Figure 2. Overall, the solutions produced by this algorithm are sufficiently accurate even under our computationally undemanding choices such as $N = 10,000$ and $T = 1,100$.

4 Stochastic versus non-stochastic simulation

In this section, we compare the stochastic- and non-stochastic-simulation versions of the Krusell-Smith method. To this purpose, we replace the procedure for simulating a panel of agents in our benchmark Krusell-Smith algorithm with a procedure for simulating the evolution of capital distribution on a grid of pre-specified points, as described in the appendix of Den Haan (2008). We outline the non-stochastic-simulation method below.

Algorithm 2 (non-stochastic simulation).

- *Step I*. Fix an initial vector of coefficients b. Generate and fix a time

have R^2 in excess of 0.999999, differ substantially in terms of the mean aggregate capital stock predicted. We have similar findings: the stochastic-simulation Krusell-Smith algorithm considered in this section yields a capital-distribution mean of 39.357, while the non-stochastic-simulation Krusell-Smith algorithm described in the next section (and also characterized by R^2 in excess of 0.9999) yields a mean of 39.037.

series of length T for the aggregate shocks. Fix the initial distribution of capital for the employed and unemployed agents on an equally spaced $1,000$-point grid over the interval $[0,100]$, i.e., $\kappa_0 = 0$ and $\kappa_j = 0.1j$, $j = 1,...,1000$.

- *Step II*. Identical to Step II of Algorithm 1.
- *Step III*. Use the individual policy rules computed in *Step II* to simulate the economy T periods forward by computing the evolution of the capital distribution on the grid, as described in Den Haan (2008), and by calculating the set of statistics m_t for each $t = 1,...,T$.
- *Steps* $IV - V$. These are identical to Steps IV-V of Algorithm 1.

Iterate on *Steps* $II - V$ until the average squared difference between $\overset{\approx}{b}$ and b is less than a given degree of precision, which we set in this case as 10^{-8}.

Den Haan (2008) proposes to compute the next-period capital distribution on the grid by solving the following non-linear problem at each grid point:

$$k'\left(x_t^{\varepsilon,j},\cdot\right) = \kappa_j, \tag{10}$$

where $x_t^{\varepsilon,j}$ is the current level of capital of agents with an employment status $\varepsilon \in \{0,1\}$ such that the future capital choice is equal to the grid point κ_j. We solve (10) by using the interpolation twice. Specifically, for each $t = 1,...,T$:

1) for given m_t and a_t, we interpolate the policy rules for employed and unemployed agents to obtain $k'(k,1,m_t,a_t)$ and $k'(k,0,m_t,a_t)$, respectively;

2) we define the inverse functions of $k'(k,1,m_t,a_t)$ and $k'(k,0,m_t,a_t)$ (i.e., we view k' as an argument, and we view k as a function of k'), and we use interpolation to restore the values of the inverse functions at each grid point κ_i.[8]

We take the initial capital distribution on the grid from Den Haan, Judd and Juillard (2008). To make this algorithm comparable to the stochastic-simulation algorithm, we use the same simulation length of $T = 1,100$, and we discard the first 100 periods when re-estimating ALM (8) on *Step IV*.

[8] A similar interpolation approach is used in Maliar and Maliar (2006) to solve for an equilibrium interest rate in Hugget's (1993) and Aiyagari's (1994) model extended to include quasi-geometric (hyperbolic) consumers.

The two interpolation steps, which are components of Algorithm 2 but are absent under Algorithm 1, are costly. As a result, the computational costs associated with the non-stochastic simulation is higher than the computational costs of the stochastic simulation. Specifically, we find that running Algorithm 1 with a panel of $10,000$ agents costs approximately the same as running Algorithm 2 with a grid of $1,000$ points, and the results are of similar accuracy. Presumably, we can reduce the cost of non-stochastic simulation by solving the non-linear problem (10) with a procedure that is more efficient than our double interpolation.[9]

Under Algorithm 2, the ALM for the bad and good aggregate states are, respectively,

$$\ln\left(K_{t+1}\right) = 0.122146 + 0.965942 \ln\left(K_t\right),$$

and

$$\ln\left(K_{t+1}\right) = 0.136272 + 0.963582 \ln\left(K_t\right).$$

The R^2 values of these two regressions were both in excess of 0.999999, and both were also higher than the R^2s produced under Algorithm 1. Again, however, this does not necessarily mean that Algorithm 2 produces more accurate solutions than Algorithm 1.

To determine the relative accuracy of Algorithm 2, we perform the same two accuracy tests on Algorithm 2 that were applied to Algorithm 1. For a simulation of $10,000$ periods with a random realization of shocks, Algorithm 2 produces an average error of 0.044%, which is smaller than the error of 0.050% generated under Algorithm 1. However, the maximum error under Algorithm 2 is 0.187%, which is somewhat larger than Algorithm 1's error of 0.156%. We do not provide a figure for the series obtained under Algorithm 2 as a result of this test, as such a figure is visually identical to Figure 1 which was obtained under Algorithm 1.

For a simulation of 200 periods with a peculiar shock sequence (100 periods of bad shocks and 100 periods of good shocks), Algorithm 2 produces average and maximum ALM errors of 0.087% and 0.182%, respectively. These are again somewhat larger than the corresponding errors generated by Algorithm 1 which are 0.062% and 0.146%. In Figure 3, we plot the aggregate capital series generated by the individual policy rule and by the ALM under

[9] Young (2008) proposes a different variant of a non-stochastic simulation procedure where the current capital is assumed to be on the grid and the next-period capital is obtained from the capital policy function. This procedure does not require an inverse and is consequently much faster.

Algorithm 2. A comparison of Figures 2 and 3 shows that unlike Algorithm 1, which generates the largest errors toward the end of the simulation, Algorithm 2 generates the largest errors around the middle of the simulation, toward the end of the bad period.

As an additional accuracy check, we compute the average and maximum Euler-equation errors for a simulation of $10,000$ periods on a random realization of shocks (see Table 1). In the benchmark case, Algorithm 2 produces slightly larger Euler-equation errors than Algorithm 1 does (see column 1). We also investigate the dependence of the Euler-equation errors on the accuracy of the individual policy rule by increasing the number of capital grid points in the individual problem from 100 to 300 (see column 2), and we study the dependence of the Euler-equation errors on the simulation length by increasing T from $1,100$ to $10,100$ (see column 3). As the table shows, these two modifications have little effect on the magnitudes of the errors. We perform additional sensitivity experiments by varying the number of agents in Algorithm 1 and the number of grid points in Algorithm 2, and we find that the Euler-equation errors are not significantly affected.

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Table 1. Euler-equation errors for a simulation of 10,000 periods on a random realization of shocks.

	100 grid points, T=1,100		300 grid points, T=1,100		100 grid points, T=10,100	
	Average	Maximum	Average	Maximum	Average	Maximum
Algorithm 1	0.0065%	0.1569%	0.0059%	0.0965%	0.0095%	0.1449%
Algorithm 2	0.0067%	0.1546%	0.0060%	0.0966%	0.0066%	0.1563%

Figure 1. Accuracy of the aggregate law of motion under Algorithm 1 (random shocks).

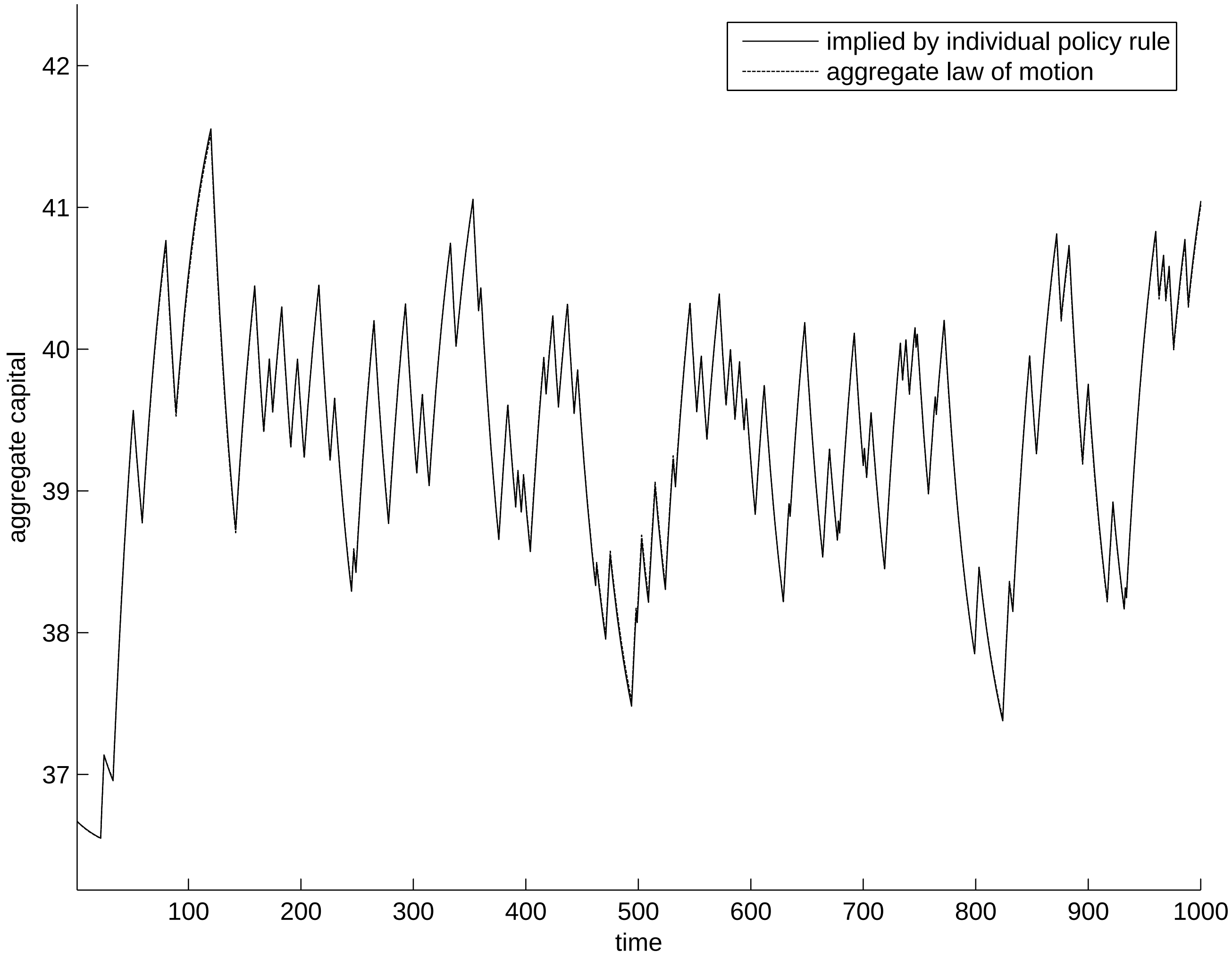


Figure 2. Accuracy of the aggregate law of motion under Algorithm 1 (peculiar shocks).

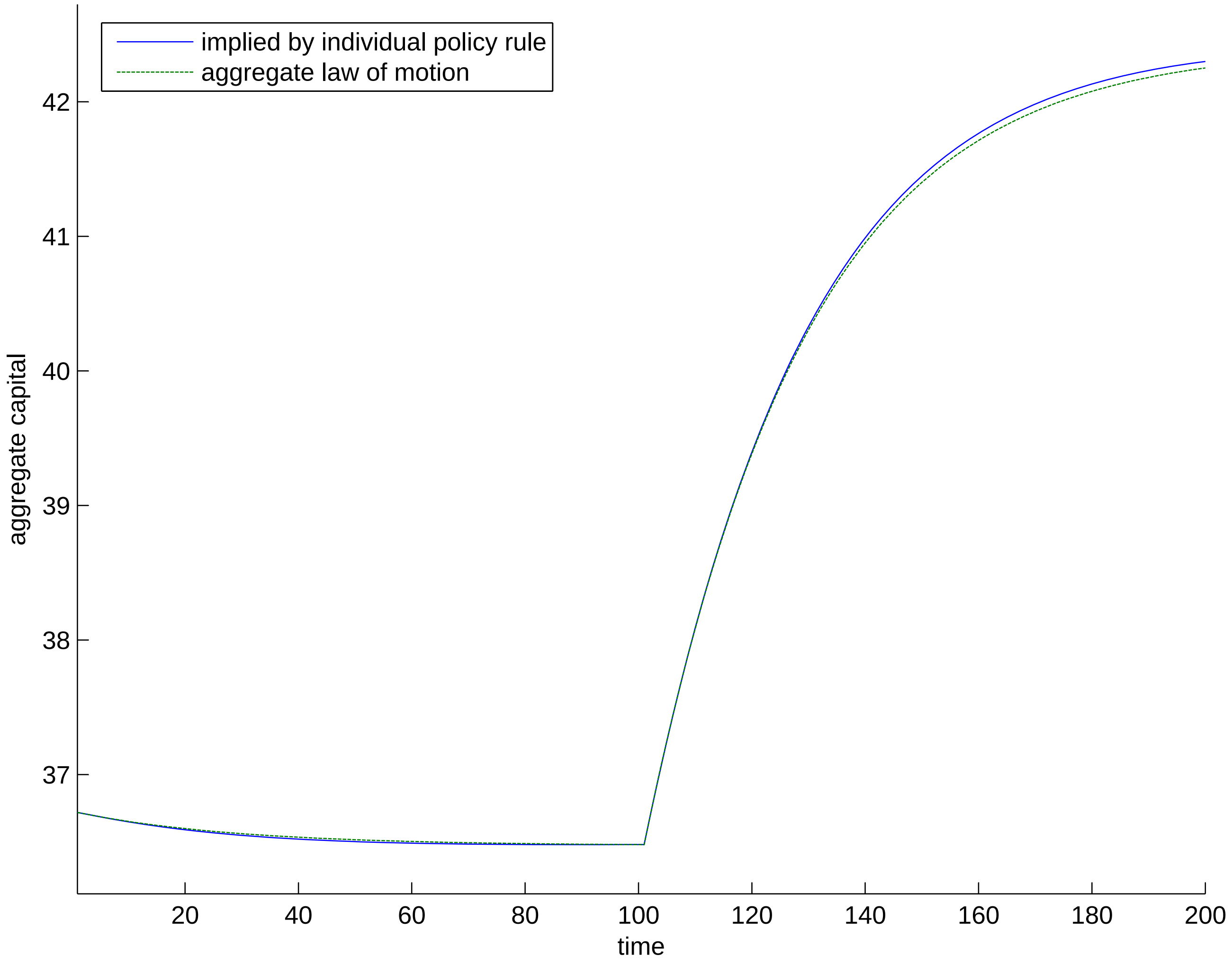
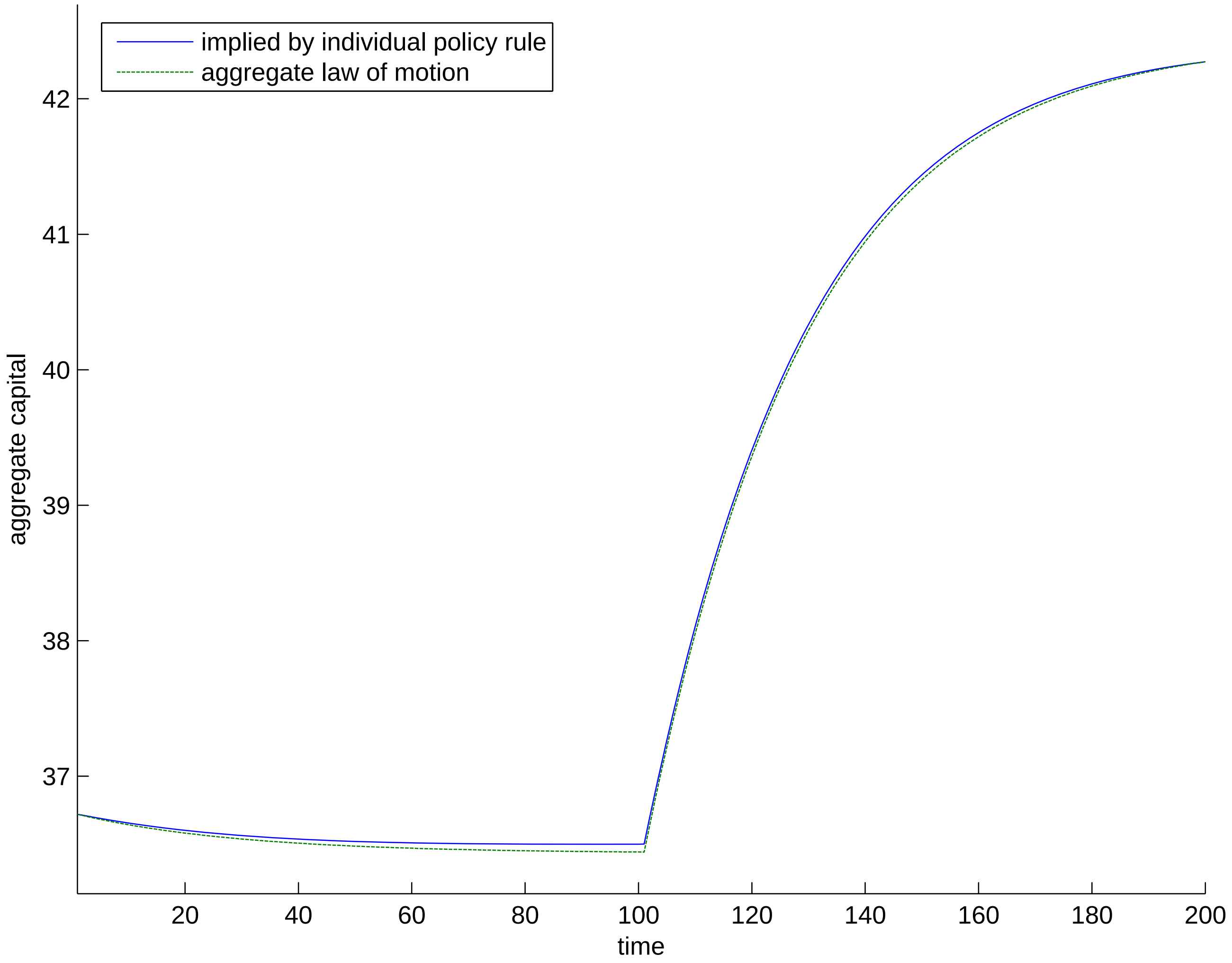


Figure 3. Accuracy of the aggregate law of motion under Algorithm 2 (peculiar shocks).


implied by individual policy rule
aggregate law of motion
aggregate capital
time