Matlab Toolbox for Solving Regular Linear Rational Expectations Models. The Algorithm

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Abstract

We present a computationally effective method of solving regular linear dynamic systems based on Schur decomposition. One of the advantage of presented method is possibility of problem dimension reduction. This allows for efficient solution of models with very large set of endogenous variables and moderate set of state variables. Presented method allows for considering models with sunspots. We also derive first differential of matrices describing model dynamics with respect to model parameters. The toolbox `lrem_solve` implements methods described in this paper.

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1 Introduction

We present a method of solving regular linear rational expectations models in general form based on generalized Schur decomposition. Such a models arises naturally, using the perturbation technique of solving dynamic economies.

Presented method is rather standard. See e.g. [2]. There are a few packages implementing this technique. This methods however differs

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substantially from existing solutions techniques. The most important differences is lack of more detailed structure of the models, we do not analyze predefined state variables explicitly. In many cases determining state variables is complicated, especially in case of large models. Additionally, existing techniques generally breaks down is the set of predefined state variables is lower, than dimension of states in the models. Such a case arises naturally in models with sunspots. Analyzing unstructurized models avoids this difficulties and allows easily for considering sunspots. We present however methods of reformulating model dynamics in terms of predefined states.

Analyzing structurized models allows for more efficient solutions in case of analyzing explicitly exogenous state variables. However computational advantage of analyzing such models in not high but introduces many difficulties. Solving models with predefined exogenous states requires solving generalized Sylvester equations. Current implementations of methods of solving such problems work only if there is exactly one solution to this equation. However generally this is not the case.

Presented technique delivers all solutions to the model or shows that there is no solution.

In order to allow analyzing large dynamic models we present a technique of reduction the problem dimension in case of very large number of endogenous variables but small or moderate number of state variables. The `lrem_solve` does not deliver completely this functionality, because such reduction technique is the most efficient in case of using available linear solvers for sparse matrices. The toolbox does not consider sparse systems.

In many cases solving rational expectations models is not the last step and they require parameter estimation, which is extremely costly. In order to reduce this cost we present how to calculate differentials of solution with respect to model parameters. The `lrem_solve` toolbox delivers efficient implementation of presented algorithm.

This paper is organized as follows: section 2 presents the problem, section 3 analyzes a matrix equation determining solution to deterministic part of the models, section 4 analyzes stochastic part of the model, in section 5 we presents the method of reducing problem dimension, section 6 delivers differentials of solution with respect to additional parameters, in section 7 we present a method of expressing model dynamics in terms of predefined set of state variables.

2 The Problem

Let us consider the following linear system

$$0 = Ay_t + By_{t+1} + CE_t y_{t+1} + V\epsilon_t + W\epsilon_{t+1} \tag{1}$$

where $y \in \mathbb{R}^m$ is a vector of control variables, $\epsilon_{t+1} \in \mathbb{R}^s$ is a vector of i.i.d. random variables normally distributed with zero mean.

We are looking for a solution in the form

$$\begin{aligned} y_t &= Ru_t + S_1\epsilon_t + S_2\omega_t \\ u_t &= Pu_{t-1} + Q_1\epsilon_t + Q_2\omega_t \end{aligned} \tag{2}$$

where $u \in \mathbb{R}^k$ is some vector of state variables and $\omega_t \in \mathbb{R}^r$ is an vector of i.i.d. random variable with zero mean independent on ϵ_t. Additionally we assume the following growth restriction:

Assumption 2.1. *We are looking for linear solutions to the system (1) such that the following growth restriction holds*

$$\lim_{t\to\infty} E_0\Big\{\xi^t y_t\Big\} = 0 \tag{3}$$

for any u_0.

Substituting (2) to (1) yields

$$\begin{aligned} 0 = {} & (AR + (B+C)RP)u_t + AS\upsilon_t + B(RQ+S)\upsilon_{t+1} \\ & + V\epsilon_t + W\epsilon_{t+1} \end{aligned}$$

this equation must be fulfilled for all u_t, ϵ_t, ω_t, hence

$$0 = AR + (B+C)RP \tag{4}$$

$$0 = AS_1 + V \qquad 0 = AS_2 \tag{5}$$

$$0 = B(RQ_1 + S_1) + W \qquad 0 = B(RQ_2 + S_2) \tag{6}$$

The transversality condition implies that

$$\lim_{t\to\infty} \xi^t RP^t = 0$$

We restrict ourselves only to regular systems, such that the matrix pair $(A, B + C)$ is regular, i.e. matrices A, $B + C$ are square and $\det(\alpha A - \beta(B + C)) \neq 0$ for some, possibly complex, α, β.

3 The matrix equation $\mathcal{A}R = \mathcal{B}RP$

In this section we briefly present a method of solving the matrix equation $\mathcal{A}R = \mathcal{B}RP$ for regular matrix pair (A, B). See [1] for further details.

In this section we are looking for matrices R and P, such that for a given square matrices $\mathcal{A}$ and $\mathcal{B}$ the equation $\mathcal{A}U = \mathcal{B}U\Sigma$ holds and the transversality condition $\lim_{t\to\infty} \xi^t RP^t = 0$ is satisfied.

Let us assume that a matrix pair $(\mathcal{A}, \mathcal{B})$ is regular[1]. Let us consider generalized Schur decomposition of the matrix pair $(\mathcal{A}, \mathcal{B})$

$$V'\mathcal{A}U = T_A \qquad\qquad V'\mathcal{B}U = T_B$$

where matrices U and V are orthogonal, the matrix T_A is quasi-upper triangular, and the matrix T_B is upper triangular. Such a decomposition always exists. Let λ_i^A, λ_i^B are i-th eigenvalues of T_A and T_B respectively. Let $\lambda_i = \lambda_i^A/\lambda_i^B$ and let λ is a set of all distinct finite eigenvalues λ_i. Let q is a size of the set λ.

Let us sort eigenvalues of T_A and T_B is such a way that all eigenvalues λ_i, such that $|\xi\lambda_i| < 1$ appears in left upper block of T_A and T_B. Then

$$\begin{aligned} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_A & T_{12}^A \\ 0 & T_{22}^A \end{bmatrix} &= \mathcal{A} \begin{bmatrix} U_1 & U_2 \end{bmatrix} \\ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_B & T_{12}^B \\ 0 & T_{22}^B \end{bmatrix} &= \mathcal{B} \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{aligned}$$

where R_A is quasi-upper triangular, R_B is upper-triangular, both matrices have the same size. This implies

$$\mathcal{A}U_1 = V_1 R_A \qquad\qquad \mathcal{B}U_1 = V_1 R_B \tag{7}$$

By assumption, the matrix R_B is invertible. Thus,

$$\mathcal{A}U_1 = \mathcal{B}U_1(R_B)^{-1}R_A$$

Then we can take $P = (R_B)^{-1}R_A$ and $R = U_1$. If matrices R, P satisfy the equation $\mathcal{A}R = \mathcal{B}RP$, then matrices $\tilde{R} = R\Xi$, $\tilde{P} = \Xi^{-1}P\Xi$ also satisfy this equation for any invertible matrix Ξ.

Theorem 3.1. $\ker BR = 0$

Proof. Let $x \in \ker B$. Then $0 = BRx = V_1 R_B x$. V_1 has full column rank, hence $R_B x = 0$. The matrix R_B is invertible, thus $x = 0$. $\square$

[1] This assumption guarantees that the Schur decomposition is numerically stable. In opposite case a matrix pair $(\mathcal{A}, \mathcal{B})$ has infinitely many eigenvalues. Small perturbations of $(\mathcal{A}, \mathcal{B})$ may drastically change matrices T_A and T_B.

4 The stochastic part

Let us consider a matrix equation $AX + B = 0$ for any matrices A, B. We are going to find all solutions to this equation. Let U is such a matrix[2] that

$$U'A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$

where A_1 is a matrix with full rank with q_1 rows. Let $U'B = \mathrm{col}(B_1, B_2)$ be the corresponding partition of the matrix $U'B$. Then

$$\begin{bmatrix} A_1 \\ 0 \end{bmatrix} X = -\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

If the matrix B_2 is not a zero matrix, then there is no solution to the equation $AX + B = 0$. Assume that this is not the case. We have

$$A_1 X + B_1 = 0;$$

If A_1 is a square matrix then A_1 is invertible and $X = A_1^{-1} B_1$ is the only solution. Let us assume that A_1 is not square and V is such a matrix that

$$A_1 V = \begin{bmatrix} A_2 & 0 \end{bmatrix}$$

where A_2 is a square matrix with q_1 rows. Then

$$\begin{bmatrix} A_2 & 0 \end{bmatrix} V'X + B_1 = 0;$$

Let $V'X = \mathrm{col}(X_1, X_2)$ be a partition of the matrix $V'X$ such that $\dim_1 X_1 = q_1$. Then

$$A_2 X_1 + B_1 = 0$$

and X_2 is any matrix. The matrix A_2 is invertible thus

$$X_1 = -A_2^{-1} B_1$$

and

$$X = V \begin{bmatrix} -A_2^{-1} B_1 \\ X_2 \end{bmatrix} = -V_1 A_2^{-1} B_1 + V_2 X_2$$

[2]To obtain the matrix U one can consider qr decomposition of the matrix A. However to obtain correct estimation of rank of the matrix A and consequently to select properly nonzero part A_1, the smallest nonzero singular value of the matrix A must be much higher than machine precision. If this is not the case, then more reliable methods are required, e.g. svd decomposition or rank revealing qr decomposition.

where X_2 is any matrix with appropriate size, and $V = [V_1, V_2]$.

Let us consider equations (5-6). Let

$$\begin{aligned} S_1 &= \tilde{S}_1 + \Phi_1 Y_1 \\ S_2 &= \Phi_1 Y_2 \end{aligned}$$

are all solutions to equation $AS_1 + V = 0$ and $AS_2 = 0$, where Y_1, Y_2 are any matrices with appropriate size, assuming that solutions exist. Then

$$0 = B \begin{bmatrix} R & \Psi_1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Y_1 \end{bmatrix} + B\tilde{S}_1 + W \equiv B\tilde{R}\tilde{Q}_1 + \tilde{W} \tag{8}$$

$$0 = B \begin{bmatrix} R & \Psi_1 \end{bmatrix} \begin{bmatrix} Q_2 \\ Y_2 \end{bmatrix} \equiv B\tilde{R}\tilde{Q}_2 \tag{9}$$

Applying again the procedure described earlier we obtain that if solutions to (8-9) exist then they take the form

$$\tilde{Q}_1 = \Xi^1 + \Psi^1 Z_1 \qquad \tilde{Q}_2 = \Psi^1 Z_2$$

where Z_1, Z_2 are any matrices with appropriate size. Hence,

$$\begin{aligned} Q_1 &= \Xi^1_1 + \Psi^1_1 Z_1 & Q_2 &= \Psi^1_1 Z_2 \\ Y_1 &= \Xi^1_2 + \Psi^1_2 Z_1 & Y_2 &= \Psi^1_2 Z_2 \end{aligned}$$

where $\Psi^1 = \mathrm{col}(\Psi^1_2, \Psi^1_2)$, $\Xi^1 = \mathrm{col}(\Xi^1_2, \Xi^1_2)$ are appropriate partition of matrices Ψ^1 and Ξ^1. Finally

$$\begin{aligned} Q_1 &= \Xi^1_1 + \Psi^1_1 Z_1 & Q_2 &= \Psi^1_1 Z_2 \\ S_1 &= \tilde{S}_1 + \Phi_1 \Xi^1_2 + \Phi_1 \Psi^1_2 Z_1 & S_2 &= \Phi_1 \Psi^1_2 Z_2 \end{aligned}$$

In this way we obtain all solutions to (5-6).

5 Problem reduction

Usually the matrix $B + C$ has large null space especially in case of large models. We can use this property to decrease computation cost of solving the problem (4). Let Q is an orthogonal matrix such that

$$Q'(B + C) = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \equiv \tilde{B}$$

and let $Q'A = \mathrm{col}(\tilde{A}_1, \tilde{A}_2) \equiv \tilde{A}$ be corresponding partition of the matrix $Q'A$. Then we have

$$\begin{aligned} \tilde{A}_1 R + \tilde{B}_1 RP &= 0 \\ \tilde{A}_2 R &= 0 \end{aligned}$$

Hence $R \in \ker \tilde{A}_2$, and there exists a T such that $R = MT$, where $M = \text{null}\, \tilde{A}_2$, and

$$\tilde{A}_1 MT + \tilde{B}_1 MTP = 0 \tag{10}$$

Theorem 5.1. *If the matrix pair $(A, B + C)$ is regular, then matrices $\tilde{A}_1 M$ and $\tilde{B}_1 M$ are square and the matrix pair $(\tilde{A}_1 M, \tilde{B}_1 M)$ is regular.*

Proof. Let $\Gamma = [\text{null}\, \tilde{A}_2, \text{range}\, \tilde{A}_2]$. Matrix Γ is orthogonal. Let us consider matrices $\tilde{A}\Gamma$ and $\tilde{B}\Gamma$. We have

$$\tilde{A}\Gamma = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \qquad \tilde{B}\Gamma = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & 0 \end{bmatrix}$$

where $\tilde{A}_{11}$ and $\tilde{B}_{11}$ are square matrices with $n_1 = \tilde{A}_{11}$ rows. Let $\alpha, \beta \in C$. We have $|\det(\alpha A - \beta(B + C))| = |\det(\alpha \tilde{A} - \beta \tilde{B})| = |\det(\alpha \tilde{A}\Gamma - \beta \tilde{B}\Gamma)| = |\det(\alpha \tilde{A}_{11} - \beta \tilde{B}_{11}||\det(\alpha \tilde{A}_{22})|$.

Assume that $\text{rank}\, \tilde{A}_2 < n - n_1$. Then $\text{rank}\, M = m > n_1$. Then, by the construction of the matrix M, the first $m - n1 > 0$ columns of the matrix $\tilde{A}_{22}$ contain only zero elements. Hence, $|\det(\tilde{A}_{22})| = 0$, and the matrix pair $(A, B + C)$ is not regular. On the other hand $\text{rank}\, \tilde{A}_2 \leq n - n_1$, because the matrix $\tilde{A}_2$ contains $n - n_1$ rows. Thus $\text{rank}\, \tilde{A}_2 = n - n_1$ and $\text{rank}\, M = n_1$. The $n \times k$ matrix M has full column rank, thus $k = n_1$ and matrices $\tilde{A}_1 M$, $\tilde{B}_1 M$ are square.

If the matrix pair $(A, B + C)$ is regular then there exist $\alpha, \beta \in C$ such that $|\det(\alpha A - \beta(B + C))| \neq 0$. Then also $|\det(\alpha \tilde{A}_{11} - \beta \tilde{B}_{11}| \neq 0$ and the matrix pair $(\tilde{A}_1 M, \tilde{B}_1 M)$ is regular. □

In this way we have proved that instead solving the problem (4) we can solve the problem (10).

6 Models dependent on parameters

Let us assume that matrices describing the model (1) depend on additional scalar parameter, θ. Then equations (4-6) take the form

$$\begin{aligned} 0 &= A(\theta)R(\theta) + (B(\theta) + C(\theta))R(\theta)P(\theta) \\ 0 &= A(\theta)S_1(\theta) + V(\theta) \\ 0 &= A(\theta)S_2(\theta) \\ 0 &= B(\theta)(R(\theta)Q_1(\theta) + S_1(\theta)) + W(\theta) \\ 0 &= B(\theta)(R(\theta)Q_2(\theta) + S_2(\theta)) \end{aligned} \tag{11}$$

and the transversality condition takes the form

$$\lim_{t \to \infty} \xi(\theta)^t R(\theta) P(\theta)^t = 0$$

Let for the basic model $\theta = 0$. Let us assume that model matrices are differentiable with respect to θ, equation (4) has solutions R and P in some neighborhood of $\theta = 0$ and these solutions are differentiable with respect to θ[3]. Then equation (14) also has solution. We are going to expand matrices R, P, S_1, S_2, Q_1, Q_2 in asymptotic series around $\theta = 0$. We already have the zero order terms.

Let $\bar{R} = R(0)$, $\bar{P} = P(0)$, $\bar{S}_1 = S_1(0)$, $\bar{S}_2 = S_2(0)$, $\bar{Q}_1 = Q_1(0)$, $\bar{Q}_2 = Q_2(0)$ solve (11). Differentiating (11) with respect to θ around $\theta = 0$ yields

$$\begin{aligned} 0 &= A'(\theta)\bar{R} + (B'(\theta) + C'(\theta))\bar{R}\bar{P} + AR'(\theta) + (B + C)R'(\theta)\bar{P} \\ &\quad + (B + C)\bar{R}P'(\theta) \end{aligned} \tag{12}$$

and

$$\begin{aligned} 0 &= A'(\theta)\bar{S}_1 + V'(\theta) + AS_1'(\theta) \\ 0 &= A'(\theta)\bar{S}_2 + AS_2'(\theta) \\ 0 &= B'(\theta)(\bar{R}\bar{Q}_1 + \bar{S}_1) + W'(\theta) + BR'(\theta)\bar{Q}_1 + B(\bar{R}Q_1'(\theta) + S_1'(\theta)) \\ 0 &= B'(\theta)(\bar{R}\bar{Q}_2 + \bar{S}_2) + BR'(\theta)\bar{Q}_2 + B(\bar{R}Q_2'(\theta) + S_2'(\theta)) \end{aligned} \tag{13}$$

where $A = A(0)$, $B = B(0)$, $C = C(0)$, $V = V(0)$, $W = W(0)$.

Equations (13) can be solved exactly in the same way as equations (5-6) using methods from the section 4[4].

Equation (12) takes the form

$$0 = \Gamma^1 + AX + (B + C)X\bar{P} + (B + C)\bar{R}Y \tag{14}$$

where $\Gamma^1 = A'(\theta)\bar{R} + (B'(\theta) + C'(\theta))\bar{R}\bar{P}$, $X = R'(\theta)$, and $Y = P'(\theta)$, with unknown matrices X and Y. There are many solutions to (14). Observe that if X is a solution to (14), then $X + \alpha\bar{R}$ is also a solution for any α. To avoid this indeterminacy we can utilize the fact, that if $R(\theta)$ is a solution to (4), then $R(\theta)U$ is also a solution for any invertible matrix U. Let $M(\theta)'$ spans the range of $R(\theta)$. Then $M(\theta)'R(\theta)$ has full row rank. Let us assume that in some neighborhood of $\theta = 0$ the matrix $M(0)'R(\theta)$ also has full row rank. Then there exists an invertible matrix $U(\theta)$ such that $M(0)'R(\theta)U(\theta) = I$ and as a new solution we can take $\tilde{R}(\theta) = R(\theta)U(\theta)U^{-1}(0)$. Then $\tilde{R}(0) = \bar{R}$, $\tilde{R}'(0) = 0$ and we can assume that $M(0)'X = 0$. Hence,

$$X = K\tilde{X}$$

[3]Generally this conditions need not be fulfilled. E.g. a model $0 = y_t + \theta\epsilon_{t+1}$ has solution only for $\theta = 0$, a model $\theta = E_t y_{t+1}$ may have solution $y_t = \theta + \sigma(\theta)\epsilon_t$, where the function $\sigma(\theta)$ is not differentiable.

[4]Generally there may be infinitely many solutions to (13), for example in case of a regular model $\theta = E_t y_{t+1}$, with one of the solution $y_t = \theta + \alpha\theta\epsilon_t$ for any α.

where K spans the null space of $\bar{R}'$. Equation (14) now takes the form

$$0 = \Gamma^1 + AK\tilde{X} + (B+C)K\tilde{X}\bar{P} + (B+C)\bar{R}Y$$

We are going to express the matrix Y in terms of $\tilde{X}$. Let matrices N, M span the null space and the range of $(B+C)'$. Then

$$0 = N'\Gamma^1 + N'AK\tilde{X} \tag{15}$$

$$0 = M'\Gamma^1 + M'AK\tilde{X} + M'(B+C)K\tilde{X}\bar{P} + M'(B+C)\bar{R}Y \tag{16}$$

Theorem 6.1. *If the matrix pair* $(A, B{+}C)$ *is regular, then* $\ker(N'AK)' = 0$.

Proof. Let $\Gamma = \mathrm{col}(N', M')$. Then Γ is square orthogonal matrix and

$$\Gamma A = \begin{bmatrix} N'A \\ \tilde{A}_2 \end{bmatrix}, \qquad \Gamma(B+C) = \begin{bmatrix} 0 \\ \tilde{B}_2 \end{bmatrix}$$

Let $\ker N'A \neq 0$. Let $\bar{N}$, $\bar{M}$ span null space and range of $(N'A)'$ and let

$$\Delta = \begin{bmatrix} \bar{N}' & 0 \\ \bar{M}' & 0 \\ 0 & I \end{bmatrix}$$

Then Δ is a square orthogonal matrix and

$$\Delta\Gamma A = \begin{bmatrix} 0 \\ \bar{A}_2 \end{bmatrix}, \qquad \Delta\Gamma(B+C) = \begin{bmatrix} 0 \\ \tilde{B}_2 \end{bmatrix}$$

Let α, *beta* are any complex scalars. Then $\det(\alpha A - \beta(B+C)) = \det(\alpha\Delta\Gamma\alpha A - \beta\Delta\Gamma\beta(B+C)) = 0$ and the matrix pair $(A, B+C)$ is not regular. Thus, $\ker(N'A)' = 0$.

Let $x \in \ker(N'AK)'$. Then $0 = K'(N'A)'x$ and $x \in \ker(N'A)'$, because $\ker K = 0$ by construction. Hence $x = 0$. □

Theorem 6.1 implies that the equation (15) always has at least one solution. Let

$$\tilde{X} = G + HZ$$

for appropriate matrices G, H and any matrix Z.

Now the equation (16) takes the form

$$0 = \Gamma_2 + M'AKHZ + M'(B+C)KHZ\bar{P} + M'(B+C)\bar{R}Y \tag{17}$$

where $\Gamma_2 = M'\Gamma^1 + M'AKG + M'(B+C)KG\bar{P}$ with unknown Y and Z. Again let matrices $\tilde{N}$, $\tilde{M}$ span the null space and the range of $(M'(B+C)\bar{R})'$. Then

$$0 = \tilde{N}'\Gamma_2 + \tilde{N}'M'AKHZ + \tilde{N}'M'(B+C)KHZ\bar{P} \tag{18}$$

and

$$0 = \tilde{M}'\Gamma_2 + \tilde{M}'M'AKHZ + \tilde{M}'M'(B+C)KHZ\bar{P} + \tilde{M}'M'(B+C)\bar{R}Y \tag{19}$$

Theorem 6.2. *If the matrix pair $(A, B+C)$ is regular, then for any Z the equation (19) has exactly one solution.*

Proof. By the construction the matrix $\tilde{M}'M'(B+C)\bar{R}$ has full row rank, and the equation (19) has at least one solution for any Z. Assume that there exist a second solution $\tilde{Y} \neq Y$. Then $\tilde{M}'M'(B+C)\bar{R}(Y-\tilde{Y}) = 0$. By the construction of the matrix $\tilde{M}$ this implies that $\tilde{M}'M'(B+C)\bar{R}(Y-\tilde{Y}) = 0$. The matrix $\tilde{M}'$ spans range of $M'(B+C)\bar{R}$, hence $M'(B+C)\bar{R}(Y-\tilde{Y}) = 0$. The matrix M' spans range of $B+C$, hence $(B+C)\bar{R}(Y-\tilde{Y}) = 0$. This contradicts the theorem 3.1. Regularity condition is required, because only in this case we can solve the model (1) using the generalized Schur decomposition. □

By the theorem 6.2 the matrix $\tilde{M}'M'(B+C)\bar{R}$ is square and invertible.

Theorem 6.3. *If the matrix pair $(A, B+C)$ is regular, then matrices $\tilde{N}'M'AKH$ and $\tilde{N}'M'(B+C)KH$ are square.*

Proof. We need to prove that $\dim_1 \tilde{N}' = \dim_2 H$ what is equivalent to $\dim_2 \operatorname{null}(\bar{R}'(B+C)'M) = \dim_2 \operatorname{null} N'AK$. By the theorem 6.2 the matrix $\tilde{M}'M'(B+C)\bar{R}$ is square and invertible, hence the matrix $Q = \bar{R}'(B+C)'M$ has full row rank and $\dim_2 \operatorname{null}(\bar{R}'(B+C)'M) = \dim_2 Q - \dim_1 Q = \dim_2 M - \dim_2 \bar{R}$.

To prove that also $\operatorname{null} N'AK$ has full rank we consider the Schur decomposition of the matrix pair $(A, B+C)$.

$$\begin{aligned} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_A & T_{12}^A \\ 0 & T_{22}^A \end{bmatrix} &= A \begin{bmatrix} U_1 & U_2 \end{bmatrix} \\ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_B & T_{12}^B \\ 0 & T_{22}^B \end{bmatrix} &= (B+C) \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{aligned} \tag{20}$$

where matrices $U = [U_1, U_2]$, $V = [V_1, V_2]$ are orthogonal, R_A is quasi-upper triangular, R_B is upper-triangular and invertable, both matrices have the same size and eigenvalues are such selected, that the growth restriction holds. Then $R = U_1\Xi$ for some invertible matrix Ξ (see section 3). In this case $K = \operatorname{null} R' = U_2$. Hence

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} T_{12}^A \\ T_{22}^A \end{bmatrix} = AK$$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} T_{12}^B \\ T_{22}^B \end{bmatrix} = (B+C)K$$

Let N spans the null space of $(B+C)'$. Because the matrix R_B is invertible, thus $N = V_2\tilde{N}$. In this way we have

$$T_{22}^A = V_2' AK \qquad\qquad T_{22}^B = V_2'(B+C)K$$

and further

$$\tilde{N}' T_{22}^A = N' AK \qquad\qquad \tilde{N}' T_{22}^B = 0$$

Regularity of the matrix pair $(A, B+C)$ requires that $\tilde{N}' T_{22}^A$ has full row rank. This ends proof that $S = N'AK$ has full row rank and $\dim_2 \operatorname{null}(N'AK) = \dim_2 S - \dim_1 S = \dim_2 K - \dim_2 N$. The second relation results from observation that $\bar{R}$ has full column rank.

Finally we have following relations: $\dim_2 N + \dim_2 M = \dim_2 B$ and $\dim_2 \bar{R} + \dim_2 K = \dim_2 B$. Hence $\dim_2 M - \dim_2 \bar{R} = \dim_2 K - \dim_2 N$ and $\dim_1 \tilde{N}' = \dim_2 H$. $\square$

Let us solve now the equation (18). We can solve this equation using vectorization technique obtaining

$$0 = \operatorname{vec}(\tilde{N}'\Gamma_2) + (I \otimes \tilde{N}' M' AKH + (\bar{P})' \otimes \tilde{N}' M'(B+C)KH) \operatorname{vec}(Z)$$

This equation can be solved using methods from the section 4, which also show whether there is any solution.

However by the theorem 6.3 we know that equation (18) is the generalized Sylvester equation, which can be solved more efficiently especially in case of large problems. Generally the equation (18) may have zero, one, or infinitely many solutions.

For small θ the modified transversality implies choosing the same eigenvalues of the matrix pair $(A(\theta), B(\theta))$ as in case $\theta = 0$.

Now we can expressed matrices describing solution to the model (4) as

$$R(\theta) \sim \bar{R} + \theta R'(\theta) \qquad\qquad P(\theta) \sim \bar{P} + \theta P'(\theta)$$
$$S_i(\theta) \sim \bar{S}_i + \theta S_i'(\theta) \qquad\qquad Q_i(\theta) \sim \bar{Q}_i + \theta Q_i'(\theta)$$

for $\theta \to 0$ and $i = 1, 2$. Such a result is very useful in estimating linear model. When there are many parameters in the model then we can repeat the procedure for all parameters.

7 Predefined state variables

Usually we would like to represent model dynamics in terms of predefined state variables. Assume that we interpret some endogenous variables as a state variables, x_t

$$x_t = Ky_t$$

which in time $t = 0$ may take any values. In generality model dynamics does not depends only on these state variables. They may appear additional state variables representing sunspots. On the other hand we would like not to introduce predefined state variables too early, because in some cases it is difficult to identify variables, which values are predetermined in given period and can take any value in period $t = 0$.

We have

$$x_t = KRu_t$$

By the assumption, all values x_t are possible, hence KR must have full row rank. Let V is an orthogonal matrix such that $KR = TV'$, where $T = [\tilde{T}, 0]$ and $\tilde{T}$ is an invertible matrix. Since KR has full row rank, such a matrix V exists. Then

$$T^{-1}x_t = V_1'u_t$$

where $V = \mathrm{col}(V_1, V_2)$ is partition of the matrix corresponding to partition of the matrix T. Let $u_t = V\tilde{u}_t$. Then

$$\tilde{T}^{-1}x_t = V_1'V\tilde{u}_t = \tilde{u}_t^1$$

where $\tilde{u}_t = [\tilde{u}_t^1, \tilde{u}_t^2]'$ is partition of the vector $\tilde{u}_t$ corresponding to partition of matrices T and V. On the other hand we have

$$\tilde{u}_t = V'PV\tilde{u}_{t-1} + V'Q\upsilon_t \equiv \tilde{P}\tilde{u}_{t-1} + \tilde{Q}\upsilon_t$$

hence

$$\begin{aligned} x_t &= \tilde{T}\tilde{P}_{11}\tilde{T}^{-1}x_{t-1} + \tilde{T}\tilde{P}_{12}\tilde{u}_{t-1}^2 + \tilde{T}\tilde{Q}^1\upsilon_t \\ \tilde{u}_t^2 &= \tilde{P}_{21}\tilde{T}^{-1}x_{t-1} + \tilde{P}_{22}\tilde{u}_{t-1}^2 + \tilde{Q}^2\upsilon_t \end{aligned}$$

where $\tilde{Q} = \mathrm{col}(\tilde{Q}^1, \tilde{Q}^2)$ is partition of the matrix $\tilde{Q}$ corresponding to partition of the matrix T and

$$\begin{bmatrix} x_t \\ \tilde{u}_t^2 \end{bmatrix} = \begin{bmatrix} \tilde{T}\tilde{P}_{11}\tilde{T}^{-1} & \tilde{T}\tilde{P}_{12} \\ \tilde{P}_{21}\tilde{T}^{-1} & \tilde{P}_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \tilde{u}_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{T}\tilde{Q}^1 \\ \tilde{Q}^2 \end{bmatrix} \upsilon_t$$

We have also

$$y_t = Ru_t + S\upsilon_t = RV \begin{bmatrix} \tilde{u}_t^1 \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t = R \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{T}^{-1} x_t \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t$$

$$= R \begin{bmatrix} V_1 \tilde{T}^{-1} & V_2 \end{bmatrix} \begin{bmatrix} x_t \\ \tilde{u}_t^2 \end{bmatrix} + S\upsilon_t$$

We do not calculate expansion of matrices describing solution to the model depending on additional parameters with predefined variables as state variables, because such a representation of model dynamics is not useful in estimation of model parameters.

8 The Toolbox

`Lrem_solve` Toolbox implements in Matlab presented algorithms. This toolbox is available from www.mathworks.com site.

9 Conclusions

We presented algorithm of analyzing general set of linear dynamic rational expectations models. We concentrated only on regular models. This excludes models with larger set of indeterminacy, e.g. models with many capital assets. Solving such models would require a method of finding ordered GUPTRI decomposition, which to our knowledge is not available yet. See [1] for further details.

References

[1] P. Kowal, *An algorithm for solving arbitrary linear rational expectations model*, working paper, June 2005.

[2] C. A. Sims, *Solving linear rational expectations models*, Computational Economics **20** (2001), 1–20.