# Solutions to Linear Expectational Difference Equations

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## 1. Introduction

Most of the issues that I address here can be found in a simple univariate example, which I adapt from Broze, Gourieroux and Szafarz (1985). Consider the following equation:

$$ {}_tw_{t+1} = aw_t + u_t, \tag{1} $$

where ${}_tw_{t+1}$ is the expectation of $w_{t+1}$ conditional on information available at time $t$. This information set includes all current and lagged values of $w_t$ and $u_t$, but no future values. $\{u_t\}$ is a first-order autoregression (AR(1)):

$$ u_t = \frac{1}{1-\rho L}\nu_t, \tag{2} $$

with $\{\nu_t\}$ a covariance stationary martingale difference sequence (MDS), that is, ${}_t\nu_{t+1} = 0$, and $|\rho| < 1$. $L$ denotes the lag operator.

To proceed, rewrite equation (1) as

$$ w_{t+1} = aw_t + u_t + \varepsilon_{t+1}, \tag{3} $$

$$ \varepsilon_{t+1} \equiv w_{t+1} - {}_t\, w_{t+1}. \tag{4} $$

Clearly $\{\varepsilon_t\}$ is an MDS; for convenience, assume that $\{\varepsilon_t\}$ is also covariance stationary. Without any restrictions, there are as many solutions to equation (3) as there are processes $\{\varepsilon_t\}$.

Let's restrict ourselves to solutions $\{w_t\}$ that satisfy

$$ \sup_{j>0} \left|{}_tw_{t+j}^2\right| < \infty, \forall j > 0. \tag{5} $$

This rules out explosive conditional means and/or variances. When $|a| < 1$,

$$ w_{t+1} = \frac{1}{1-aL}\left(u_t + \varepsilon_{t+1}\right) + a_0a^{t+1}, \tag{6} $$

for any constant $a_0$ and covariance stationary MDS $\{\varepsilon_t\}$, so that there are still infinitely many solutions. Assuming that the sequence begins with $w_0$ only pins down $a_0$; similarly, if we restrict ourselves to covariance stationary solutions, we only need

$a_0 = 0$.

I will treat the knife-edge case of $|a| = 1$ as the limit of the case where $|a| = 1$. Now let $c = 1/a$, so that $w_t = -cu_t + c_t w_{t+1}$. When $|a|$ is greater than 1 the stable solution can be found by solving (1) "forward:"

$$w_t = -c \sum_{j=0}^{\infty} c^j{}_t u_{t+j} \tag{7}$$

Given that $\{u_t\}$ is an AR(1) process, (7) becomes

$$w_t = d(L)\,\nu_t, \tag{8}$$

$$d(L) = \frac{-c}{1-c\rho}\frac{1}{1-\rho L}, \tag{9}$$

so that:

$$\varepsilon_{t+1} = \frac{-c}{1-c\rho}\nu_{t+1}. \tag{10}$$

It should not be surprising that (7) is the unique stable solution when $|a| > 1$; Gourieroux, Laffont and Monfort (1982) show this formally. Note that when $|a| > 1$, $w_t$ and $\varepsilon_t$ must lie in the space spanned by current and lagged values of $\nu_t$, which I will denote as $H_\nu(t)$, but when $|a| < 1$, there are many spaces within which $\varepsilon_t$ can reside, even if $a_0 = 0$.

Finally, consider what happens if we fix $w_0$. When $|a| > 1$, $d(L)\,\nu_0$ will equal an arbitrary $w_0$ only by chance; when there are unstable roots, there will be no solution that satisfies both a stability condition and an arbitrary initial condition. This contrasts with the case where $|a| < 1$; there we found that there were infinitely many stable solutions compatible with the initial condition.

## 2. The General Multivariate Case and its Solution

### 2.1 The Canonical Problem

Nearly all the intuition of the univariate example extends to the general multivariate

case. My approach most closely follows that of Klein (1998) and Sims (1997), both of which build off the work of Blanchard and Kahn (1980). I also incorporate elements of the MDS approaches used by Broze, Gourieroux and Szafarz (1985, 1995), Farmer (1993) and Farmer and Guo (1994), as well as insights from King, Plosser and Rebelo (1987).[2] For the most part, my approach will be pragmatic and focused; readers interested in generality and rigor are directed to the works listed above.

Consider the following system:

$$\mathbf{A_0}_t\mathbf{x}_{t+1} = \mathbf{A_1}\mathbf{x}_t, \quad t = 0, 1, 2... \tag{11}$$

$$\boldsymbol{\varepsilon}_{t+1} \equiv \mathbf{R}\left[\mathbf{x}_{t+1} -_t \mathbf{x}_{t+1}\right] \; given, \quad t = 0, 1, 2... \tag{12}$$

$$\boldsymbol{\varepsilon}_0 \equiv \mathbf{R}\mathbf{x}_0 \; given, \tag{13}$$

where: $\mathbf{x}_t$ is an $(n \times 1)$ random vector; $\mathbf{A_0}$ and $\mathbf{A_1}$ are $(n \times n)$ matrices; $\{\boldsymbol{\varepsilon}_{t+1}\}_{t=0}^{\infty}$ is a covariance stationary MDS; and $\mathbf{R}$ is a full-rank $(n_2 \times n)$ matrix, with $n_2 \leq n$. My goal is to find the stable—in the sense of equation (5)—covariance stationary processes $\{\mathbf{x}_{t+1}\}_{t=0}^{\infty}$ that satisfy equations (11) through (13).[3] This will consist in large part in finding the process for:

$$\mathbf{e}^{\mathbf{x}}_{t+1} \equiv \mathbf{x}_{t+1} -_t \mathbf{x}_{t+1}. \tag{14}$$

As the previous section suggests, $\mathbf{e}^{\mathbf{x}}_{t+1}$ need not always lie in $H_{"}(t)$.

My approach has four steps:

1. Transform the system given by equation (11), ignoring for the moment the side constraints given by equations (12) and (13).
2. Find the set of solutions to the (unrestricted) transformed system.

[2] Extensions, alternative approaches, and useful references also appear in Anderson et al. (1995), Binder and Pesaran (1995, 1997), Hamilton and Whiteman (1985), King and Watson (1997a, 1997b) and Zadrozny (1997).

[3] Strictly speaking, our solution will not be covariance stationary unless the covariance matrix of $\boldsymbol{\varepsilon}_0$ is suitably restricted.

3. "Reverse-transform" this solution into the format of the original system.
4. Working in the original format, impose the side constraints.

### 2.2 Transforming the System

The first step in solving the system is to find the real generalized Schur (or QZ) decomposition of $\mathbf{A_0}$ and $\mathbf{A_1}$.[4] The real decomposition, however, is most easily understood in contrast to the complex QZ decomposition, and so I discuss that first. The complex QZ decomposition consists of $(n \times n)$ matrices $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Z}}$ such that

$$\tilde{\mathbf{Q}}\mathbf{A_0}\tilde{\mathbf{Z}} = \tilde{\mathbf{S}} \; is \; lower \; triangular, \tag{15}$$

$$\tilde{\mathbf{Q}}\mathbf{A_1}\tilde{\mathbf{Z}} = \tilde{\mathbf{T}} \; is \; lower \; triangular. \tag{16}$$

In addition, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Z}}$ are *unitary*, in that

$$\tilde{\mathbf{Q}}^{\mathbf{T}}\tilde{\mathbf{Q}} = \tilde{\mathbf{Z}}^{\mathbf{T}}\tilde{\mathbf{Z}} = \mathbf{I}_n, \tag{17}$$

where "$\mathbf{T}$ " denotes transposition with complex conjugation. Let $\lambda_i$ denote the ratio of the $i$th diagonal elements of $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$**,** namely

$$\lambda_i \equiv \frac{\tilde{t}_{ii}}{\tilde{s}_{ii}}. \tag{18}$$

These $\lambda_i$ are called the *generalized eigenvalues* of the *matrix pencil* $\lambda\mathbf{A_0} - \mathbf{A_1}$, and if $\mathbf{A_0}$ is the identity matrix, the $\{\lambda_i\}$ are just the eigenvalues of $\mathbf{A_1}$.[5] When $\mathbf{A_0}$ is singular, some of its diagonal elements equal zero, in which case we treat the relevant $\lambda_i$ as infinite. It turns out that there exists a complex QZ decomposition for any ordering of the $\{\lambda_i\}$.

As long as the generalized eigenvalues are ordered so that members of conjugal pairs appear consecutively, every complex QZ decomposition of our system is ac-

[4]See Anderson et al. (1995) and Klein (1998) for more discussion of the QZ decomposition. In brief, the QZ decomposition is preferable to the more familiar eigenvalue-eigenvector decomposition because: (1) the QZ decomposition exists more generally, including when $\mathbf{A_0}$ is singular; and (2) the QZ decomposition is stabler numerically.

[5]Technically, the generalized eigenvalues are $\{\lambda_i : |\mathbf{A_1} - \lambda_i\mathbf{A_0}| = 0\}$.

companied by a real one.[6] In particular, there are real unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that

$$\mathbf{QA_0Z} = \mathbf{S}\ is\ real{-}valued\ and\ lower\ block\ triangular, \tag{19}$$

$$\mathbf{QA_1Z} = \mathbf{T}\ is\ real{-}valued\ and\ lower\ block\ triangular. \tag{20}$$

Let $m_1$ denote the number of generalized eigenvalues that lie outside the unit circle—that is, are unstable—and let $m_2 = n - m_1$ denote the number that lie on or within—that is, are stable.[7] Let's assume that

$$m_2 \geq n_2. \tag{21}$$

In general, systems with $m_2 < n_2$ have no solution.

The next step is to create the transformed variables:

$$\begin{pmatrix} \mathbf{q}_t \\ \mathbf{y}_t \end{pmatrix} = \mathbf{Z^T x}_t, \tag{22}$$

where $\mathbf{q}_t$ is $(m_1 \times 1)$ and $\mathbf{y}_t$ is $(m_2 \times 1)$. It follows that pre-multiplying both sides of equation (11) by $\mathbf{Q}$ yields:

$$\mathbf{S}\begin{pmatrix} {}_t\mathbf{q}_{t+1} \\ {}_t\mathbf{y}_{t+1} \end{pmatrix} = \mathbf{T}\begin{pmatrix} \mathbf{q}_t \\ \mathbf{y}_t \end{pmatrix}, \tag{23}$$

Since $\mathbf{Z^T}$ and $\mathbf{Q}$ have full rank, the original and transformed systems have the same stability properties.

Now let's assume that $\mathbf{S}$ and $\mathbf{T}$ are ordered so that the $m_1$ unstable generalized eigenvalues appear first, so that upon partitioning:

$$\begin{bmatrix} \mathbf{S_{11}} & \mathbf{0} \\ \mathbf{S_{21}} & \mathbf{S_{22}} \end{bmatrix}\begin{pmatrix} {}_t\mathbf{q}_{t+1} \\ {}_t\mathbf{y}_{t+1} \end{pmatrix} = \begin{bmatrix} \mathbf{T_{11}} & \mathbf{0} \\ \mathbf{T_{21}} & \mathbf{T_{22}} \end{bmatrix}\begin{pmatrix} \mathbf{q}_t \\ \mathbf{y}_t \end{pmatrix}. \tag{24}$$

It is this lower triangular system that we will solve first.

[6] This also requires $\mathbf{A_0}$ and $\mathbf{A_1}$ to be real-valued, as was assumed above.

[7] In the analysis below, I will treat the generalized eigenvalues that lie on the unit circle as limiting cases of the generalized eigenvalues that lie strictly inside.

### 2.3 Solving the Transformed System

As one might suspect, the matrix pencils $\lambda \mathbf{S_{jj}} - \mathbf{T_{jj}},\ j = 1, 2$, inherit the stability properties of the generalized eigenvalues used to partition $\mathbf{S}$ and $\mathbf{T}$. This means that the upper block of equations in (24) will explode unless it is solved forward:

$$\mathbf{q}_t = \mathbf{T_{11}^{-1}} \mathbf{S_{11}}\, {}_t\mathbf{q}_{t+1}, \tag{25}$$

so that

$$\mathbf{q}_t = \mathbf{0}. \tag{26}$$

In contrast, the lower block of equations is stable, in that $\lim_{j \to \infty} \left[\mathbf{S_{22}^{-1}} \mathbf{T_{22}}\right]^j = \mathbf{0}$.[8] In the absence of any additional constraints, this implies that the lower block of equations can support any solution of the form:

$$\mathbf{y}_t = \sum_{j=0}^{t-1} \left[\mathbf{S_{22}^{-1}} \mathbf{T_{22}}\right]^j \mathbf{e}^{\mathbf{y}}_{t-j} + \left[\mathbf{S_{22}^{-1}} \mathbf{T_{22}}\right]^t \mathbf{y_0}, \tag{27}$$

where $\left\{\mathbf{e}^{\mathbf{y}}_t\right\}$ is an arbitrary covariance stationary MDS, and $\mathbf{y_0}$ is $(m_2 \times 1)$.

Putting it all together, we can rewrite equation (24) as

$$\begin{pmatrix} \mathbf{q}_{t+1} \\ \mathbf{y}_{t+1} \end{pmatrix} = \mathbf{E} \begin{pmatrix} \mathbf{q}_t \\ \mathbf{y}_t \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{e}^{\mathbf{y}}_{t+1} \end{pmatrix}, \tag{28}$$

$$\mathbf{E} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S_{22}^{-1}} \mathbf{T_{22}} \end{bmatrix}, \tag{29}$$

with $\mathbf{q}_0 = \mathbf{0}$. Since all of the eigenvalues of the matrix $\mathbf{E}$ lie inside the unit circle, $\left\{ \begin{bmatrix} \mathbf{q}_t^{\mathbf{T}} & \mathbf{y}_t^{\mathbf{T}} \end{bmatrix}^{\mathbf{T}} \right\}$ is a covariance stationary VAR(1).

### 2.4 Recovering the Original System

To recover the original system, multiply both sides of equation (28) by $\mathbf{Z}$ to get

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{F}\mathbf{x}_t + \begin{bmatrix} \mathbf{Z_1} & \mathbf{Z_2} \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{e}^{\mathbf{y}}_{t+1} \end{pmatrix}, \\ &= \mathbf{F}\mathbf{x}_t + \mathbf{Z_2}\mathbf{e}^{\mathbf{y}}_{t+1}, \end{aligned} \tag{30}$$

$$\mathbf{F} \equiv \mathbf{Z}\mathbf{E}\mathbf{Z}^{\mathbf{T}}, \tag{31}$$

[8] Under regularity conditions, $\mathbf{T_{11}}$ and $\mathbf{S_{22}}$ are always invertible. See Klein (1998) for details.

where $\mathbf{Z_1}$ and $\mathbf{Z_2}$ are a partition of $\mathbf{Z}$, so that $\mathbf{Z_2}$ is $(n \times m_2)$. We recover initial values in a similar manner:

$\mathbf{x}_0 = \mathbf{Z_2 y}_0$.

In the special case where $\mathbf{A_0} = \mathbf{I}_n$, we also have

$$\mathbf{x}_{t+1} = \mathbf{A_1 x}_t + \mathbf{Z_2 e}^{\mathbf{y}}_{t+1}. \tag{32}$$

To see that (30) and (32) are equivalent, first note that when $\mathbf{A_0} = \mathbf{I}_n$, we can construct a QZ decomposition with $\mathbf{Q} = \mathbf{Z^T}$, so that[9]

$$\mathbf{A_1} = \mathbf{ZTZ^T}, \tag{33}$$

$$\mathbf{S} = \mathbf{I}_n. \tag{34}$$

Using these results, our work in section 2.3 implies that[10]

$$\begin{aligned} \mathbf{A_1 x}_t &= \mathbf{ZT} \begin{pmatrix} \mathbf{q}_t \\ \mathbf{y}_t \end{pmatrix} \\ &= \mathbf{ZE} \begin{pmatrix} \mathbf{0} \\ \mathbf{y}_t \end{pmatrix} \\ &= \mathbf{F x}_t. \end{aligned} \tag{35}$$

## 2.5 Imposing the Side Constraints

Until now, we have done nothing to pin down $\mathbf{e}^{\mathbf{y}}_t$ or $\mathbf{y}_0$. To do this, we first impose the side constraints given by equations (12) and (13).[11] In particular, equations (12) and (30) imply

$$\mathbf{R}\left[\mathbf{x}_{t+1} -_t \mathbf{x}_{t+1}\right] = \mathbf{Z_{22}} \mathbf{e}^{\mathbf{y}}_{t+1} = \boldsymbol{\varepsilon}_{t+1}, \tag{36}$$

[9] Equation (33) is also known as the real Schur decomposition of $\mathbf{A_1}$. For more discussion, see Anderson et al. (1996).

[10] When $\mathbf{A_0} = \mathbf{I}_n$, one can, with the appropriate substitutes for $\mathbf{Z}$ and $\mathbf{T}$, apply to the work in section 2.3 the more familiar eigenvalue-eigenvector decomposition utilized by Blanchard and Kahn (1980).

[11] Note that each of the side constraints is a *stochastic* restriction, expressed in terms of stochastic processes. This means, for example, that equation (12) gives $n_2 \left(n_2 + 1\right)/2$ of the $m_2 \left(m_2 + 1\right)/2$ non-zero elements of $\mathbf{e}^{\mathbf{y}}_t$'s covariance matrix.

$$\mathbf{Z_{22}} \equiv \mathbf{R}\mathbf{Z_2}. \tag{37}$$

The solution to (36) can written as

$$\mathbf{e}_t^{\mathbf{y}} = \mathbf{e}_{\mathbf{b}t}^{\mathbf{y}} + \mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\mathbf{d}_t, \tag{38}$$

where $\mathbf{e}_{\mathbf{b}t}^{\mathbf{y}}$ is a basic solution to (36), $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}$ is an $(m_2 \times m_3)$ basis for the null space of $\mathbf{Z_{22}}$, and $\{\mathbf{d}_t\}$ is an $(m_3 \times 1)$ covariance stationary MDS. Noting that $\mathbf{Z_{22}}$ is $(n_2 \times m_2)$, I will assume that it is of rank $n_2$, so that $m_3 = m_2 - n_2$.[12] Farmer (1993) makes a similar decomposition under a slightly different framework. Borrowing his terminology, one can interpret $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\mathbf{d}_t$ as a vector of "sunspots."

The general approach for finding $\mathbf{e}_{\mathbf{b}t}^{\mathbf{y}}$ is to set $m_3$ of its elements to $0$, and solve for the remaining $n_2$; see Gill, Murray and Wright (1991) for more details. It turns out that $\mathbf{e}_{\mathbf{b}t}^{\mathbf{y}}$ can be written as

$$\mathbf{e}_{\mathbf{b}t}^{\mathbf{y}} = \mathbf{D}\boldsymbol{\varepsilon}_t, \tag{39}$$

with the elements of the $(m_2 \times n_2)$ matrix $\mathbf{D}$ complicated functions of the elements of $\mathbf{Z_{22}}$. While the choice of the $m_3$ zero elements—and thus $\mathbf{D}$—is rarely unique, once they have been selected, $\mathbf{e}_{\mathbf{b}t}^{\mathbf{y}}$ is uniquely determined through $\mathbf{D}$. In practice, I set $\mathbf{D}$ and $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}$ with functions provided in GAUSS, and keep them constant over time.[13] While simple, this approach is somewhat arbitrary: since any two basic solutions differ only by an element of $\mathbf{Z_{22}}$'s null space, for a given process $\{\boldsymbol{\varepsilon}_t\}$ there are usually many pairs of processes $\{\mathbf{d}_t\}$ and matrices $\mathbf{D}$ that can generate a

[12] If $r = \text{rank}(\mathbf{Z_{22}}) < n_2$, equation (36) will have a solution only if $\boldsymbol{\varepsilon}_{t+1} = \mathbf{Z}_{\mathbf{22}}^{\mathbf{b}}\boldsymbol{\omega}_{t+1}$, where $\mathbf{Z}_{\mathbf{22}}^{\mathbf{b}}$ is a basis for the column space of $\mathbf{Z_{22}}$ and $\{\boldsymbol{\omega}_t\}$ is an $(r \times 1)$ covariance stationary MDS. As Sims (1997) points out, this immediately implies that the covariance matrix of $\boldsymbol{\varepsilon}_{t+1}$ is singular, which violates the presumption that $\{\boldsymbol{\varepsilon}_{t+1}\}$ can be chosen arbitrarily.

[13] In setting $\mathbf{D}$, I generally use the QR factorization. In particular, $\mathbf{Z_{22}}\mathbf{P} = \hat{\mathbf{Q}}\begin{bmatrix} \mathbf{R_1} & \mathbf{R_2} \end{bmatrix}$, where: $\mathbf{P}$ is an $(m_2 \times m_2)$ (unitary) permutation matrix; $\hat{\mathbf{Q}}$ is $(n_2 \times n_2)$ and unitary; $\mathbf{R_1}$ is $(n_2 \times n_2)$ and nonsingular; and $\mathbf{R_2}$ is $(n_2 \times m_3)$. With the QR factorization, $\mathbf{D} = \mathbf{P}\begin{bmatrix} \mathbf{R_1}^{-1}\hat{\mathbf{Q}}^{\mathbf{T}} \\ \mathbf{0} \end{bmatrix}$, with the zero matrix of dimension $(m_3 \times n_2)$. (A derivation appears in Gill, Murray and Wright (1991).) All of the relevant matrices are returned by the GAUSS function QQRE.

given $\{\mathbf{e}_t^{\mathbf{y}}\}$. On the other hand, as long as $\{\mathbf{d}_t\}$ is selected as the solution to some optimization problem, the choice of $\mathbf{D}$ and $\mathbf{Z_{22}^N}$ amounts to nothing more than a change of basis—I return to this point in Section 3.1 below.

The exception is the saddle-path case, $m_2 = n_2$, where $\mathbf{Z_{22}^N d}_t \equiv \mathbf{0}$ and $\mathbf{D} = \mathbf{Z_{22}^{-1}}$, so that $\mathbf{e}_t^{\mathbf{y}} = \mathbf{e}_{\mathbf{b}t}^{\mathbf{y}}$.

Combining equations (30) and (38) and (39) yields our solution:

$$\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t + \mathbf{H}\begin{pmatrix} \mathbf{d}_{t+1} \\ \boldsymbol{\varepsilon}_{t+1} \end{pmatrix}, \tag{40}$$

$$\mathbf{H} \equiv \mathbf{Z_2}\begin{bmatrix} \mathbf{Z_{22}^N} & \mathbf{D} \end{bmatrix}, \tag{41}$$

with $\mathbf{x}_0$ satisfying

$$\mathbf{x}_0 = \mathbf{H}\begin{pmatrix} \mathbf{d}_0 \\ \boldsymbol{\varepsilon}_0 \end{pmatrix}. \tag{42}$$

## 3. Extensions and Interpretatiton

### 3.1 Uniqueness with Sunspots

Recalling equation (40), the solution to our system is the process for

$$\mathbf{e}_t^{\mathbf{x}} = \mathbf{Z_2}\begin{bmatrix} \mathbf{Z_{22}^N} & \mathbf{D} \end{bmatrix}\begin{pmatrix} \mathbf{d}_t \\ \boldsymbol{\varepsilon}_t \end{pmatrix}, \tag{43}$$

with $\boldsymbol{\varepsilon}_t$ taken as given. Except in the saddle-path case, where $\mathbf{Z_{22}^N}$ and $\mathbf{d}_t$ vanish, this solution is not unique. The first step in handling the multiplicity of solutions is to make sure that we have identified all of them. The relevant task is finding the set of $\left\{\mathbf{D}, \mathbf{Z_{22}^N}, \{\mathbf{d}_t\}\right\}$ that comprise a basis for $\{\mathbf{e}_t^{\mathbf{y}}\}$.[14]

As mentioned in section 2.3, this boils down to finding a basis for all the stationary MDS processes $\{\mathbf{d}_t\}$, taking as given arbitrary choices of $\mathbf{D}$ and $\mathbf{Z_{22}^N}$. Put differently, if $\left\{\widehat{\mathbf{D}}, \widehat{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}, \left\{\widehat{\mathbf{d}}_t\right\}\right\}$ yield the process $\{\mathbf{e}_t^{\mathbf{P}}\}$, then for the alternate matrices $\left\{\widetilde{\mathbf{D}}, \widetilde{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}\right\}$, there exists a process $\left\{\widetilde{\mathbf{d}}_t\right\}$ that generates $\{\mathbf{e}_t^{\mathbf{P}}\}$ as well.

[14] Note that in searching for $\{\mathbf{d}_t\}$, one is looking across *processes*.

To see that $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}$ can be chosen arbitrarily, consider two alternative bases for $\mathbf{Z_{22}}$'s null space, $\widehat{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}$ and $\widetilde{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}$. These two bases are related by

$$\widetilde{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}} = \widehat{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}\mathbf{M}, \tag{44}$$

with $\mathbf{M}$ $(m_3 \times m_3)$ and full-rank. It immediately follows that

$$\widehat{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}\mathbf{d}_t = \widetilde{\mathbf{Z}}_{\mathbf{22}}^{\mathbf{N}}\left[\mathbf{M}^{-1}\mathbf{d}_t\right], \tag{45}$$

verifying the claim.

To see that $\mathbf{D}$ can be chosen arbitrarily as well, consider two candidate matrices, $\widehat{\mathbf{D}}$ and $\widetilde{\mathbf{D}}$. Now by construction

$$\mathbf{Z_{22}}\widehat{\mathbf{D}}\mathbf{I}_{n_2} - \mathbf{Z_{22}}\widetilde{\mathbf{D}}\mathbf{I}_{n_2} = \mathbf{I}_{n_2} - \mathbf{I}_{n_2}, \tag{46}$$

so that

$$\mathbf{Z_{22}}\left[\widehat{\mathbf{D}} - \widetilde{\mathbf{D}}\right] = \mathbf{0}. \tag{47}$$

But this means that column by column $\widehat{\mathbf{D}}$ and $\widetilde{\mathbf{D}}$ differ only by elements of $\mathbf{Z_{22}}$'s null space, so that

$$\widehat{\mathbf{D}} - \widetilde{\mathbf{D}} = \mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\mathbf{N}_1, \tag{48}$$

with $\mathbf{N}_1$ $(m_3 \times n_2)$. Consider two candidate processes $\left\{\widehat{\mathbf{d}}_t\right\}$ and $\left\{\widetilde{\mathbf{d}}_t\right\}$, with

$$\widetilde{\mathbf{d}}_t - \widehat{\mathbf{d}}_t = \mathbf{N}_1\boldsymbol{\varepsilon}_t. \tag{49}$$

Then

$$\widehat{\mathbf{D}}\boldsymbol{\varepsilon}_t + \mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\widehat{\mathbf{d}}_t = \widetilde{\mathbf{D}}\boldsymbol{\varepsilon}_t + \mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\widetilde{\mathbf{d}}_t, \tag{50}$$

which verifies the second claim.

It is useful to decompose $\mathbf{d}_t$ as

$$\mathbf{d}_t = \mathbf{N}_1\boldsymbol{\varepsilon}_t + \mathbf{d}_{2t}, \tag{51}$$

with $\mathbf{d}_{2t}$ orthogonal to $\boldsymbol{\varepsilon}_t$. ($\mathbf{N}_1$ is now arbitrary.) Typically, one can identify $\mathbf{d}_{2t}$ up only to a Choleski decomposition. This yields

$$\mathbf{d}_{2t} \equiv \mathbf{N}_2\boldsymbol{\delta}_t, \tag{52}$$

where $\mathbf{N}_2$ is $(m_3 \times m_3)$ and upper triangular, and the variance-covariance matrix of $\boldsymbol{\delta}_t$ is the identity matrix. Similarly, one can write $\boldsymbol{\varepsilon}_t$ as

$$\boldsymbol{\varepsilon}_t \equiv \mathbf{N}_0 \boldsymbol{\omega}_t, \tag{53}$$

where $\mathbf{N}_0$ is $(n_2 \times n_2)$ and upper triangular, and the variance-covariance matrix of $\boldsymbol{\omega}_t$ is the identity matrix. This yields

$$\begin{pmatrix} \mathbf{d}_t \\ \boldsymbol{\varepsilon}_t \end{pmatrix} = \mathbf{N} \begin{pmatrix} \boldsymbol{\delta}_t \\ \boldsymbol{\omega}_t \end{pmatrix}, \tag{54}$$

$$\mathbf{N} \equiv \begin{bmatrix} \mathbf{N}_2 & \mathbf{N}_1\mathbf{N}_0 \\ \mathbf{0} & \mathbf{N}_0 \end{bmatrix}. \tag{55}$$

Since, in most applications, one must pick $\boldsymbol{\varepsilon}_t$ as well as $\mathbf{d}_t$, the relevant problem is usually finding $\mathbf{N}$.[15] Once one has picked $\mathbf{N}$, however, one has completely specified the stochastic part of our linear system.

Two special cases deserve mention. The first is the saddle path case, $m_2 = n_2$, where $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}\mathbf{d}_t \equiv \mathbf{0}$, $\mathbf{D} = \mathbf{Z}_{\mathbf{22}}^{-1}$ and $\mathbf{N} = \mathbf{N}_0$. The second case is "full indeterminacy," where $m_2 = n$. In this case, once can set $\begin{bmatrix} \mathbf{Z}_{\mathbf{22}}^{\mathbf{N}} & \mathbf{D} \end{bmatrix} = \mathbf{I}_{m_2}$, as there are no restrictions whatsoever on $\mathbf{e}_t^{\mathbf{y}}$.

### 3.2 Sunspots and Exogenous Shocks

In the previous section we showed that as long as $\mathbf{N}$ is picked to satisfy some criterion, the choice of $\mathbf{D}$ and $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}$ amounts to little more than a change of basis. This leaves us with the problem with picking $\mathbf{N}$. To do this, we need $m_2$ *stochastic* restrictions. Return to the notation of the previous section, so that $\mathbf{N}$ can be decomposed as in equation (55). Under our construction, the lower right corner of corner of $\mathbf{N}$, namely the matrix $\mathbf{N}_0$ is given by the economic theory behind the side constraint (36). We need $m_3$ additional stochastic restrictions to identify $\mathbf{N}_1$ and $\mathbf{N}_2$.

[15] One can utilize non-triangular factorizations of $\mathbf{N}$ as well, as long as one imposes enough restrictions to achieve identification. (One safe generalization is to let $\mathbf{N}$ be block upper triangular, with restrictions in $\mathbf{N}_0$ and $\mathbf{N}_2$.)

Mathematically, the "theoretical" restrictions given by (36) and the restrictions used to identify "sunspots" are equivalent. As long as $\mathbf{N}$ is picked to satisfy some criterion, one is free not only to choose $\mathbf{D}$ and $\mathbf{Z}_{\mathbf{22}}^{\mathbf{N}}$, but to pick $n_2$ as well, subject only to $n_2 \leq m_2$. Upon reflection, this is not suprising. Equation (30) shows that the underlying problem is to identify the $m_2$ elements of $\mathbf{e}_t^{\mathbf{y}}$. The question then becomes, as Klein (1998) points out, of how to interpret the restrictions one places on $\mathbf{e}_t^{\mathbf{y}}$. For example, when $n_2 < m_2$, one must supplement the $n_2$ restrictions in equation (36) with $m_3$ restrictions for identifying $\mathbf{d}_t$. One could just as easily add $m_3$ rows to the system in (36).

Leeper and Sims (1994) and Klein (1998) suggest the strategy of adding or removing equations from equation (36) so that $n_2$ always equals $m_2$. The advantage of my approach is that it explicitly distinguishes between those restrictions explicitly given by economic theory, and those used to achieve econometric—or calibration—identification. For example, in the basic stochastic growth model, innovations to the capital stock are identically zero—next period's capital stock is known today—and innovations to the technology process are exogenous. If these two theoretical restrictions do not pin down $\mathbf{e}_t^{\mathbf{y}}$, one could argue, as do Farmer and Guo (1994), that the stochastic growth model supports sunspots. My approach clearly identifies the set of permissable "sunspot" processes.

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