Does the Time-Consistency Problem Explain the Behavior of Inflation in the United States?

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1. The Modified Barro-Gordon Model

1.1. The Model

The model extends Barro and Gordon's (1983) by allowing the natural rate of unemployment to follow a more general autoregressive process that contains a unit root and by introducing control errors for inflation. The actual unemployment rate U_t fluctuates around the natural rate U_t^n in response to deviations of the actual inflation rate π_t from the expected rate π_t^e:

$$U_t = U_t^n - \alpha(\pi_t - \pi_t^e), \tag{1}$$

where $\alpha > 0$. The natural rate fluctuates over time in response to real shocks according to

$$U_t^n - U_{t-1}^n = \lambda(U_{t-1}^n - U_{t-2}^n) + \varepsilon_t, \tag{2}$$

where $1 > \lambda \geq 0$ and ε_t is iid normal with mean zero and standard deviation σ_ε.

At the beginning of each period $t = 0, 1, 2, ...$, after private agents have formed their expectation π_t^e but prior to the realization of the shock ε_t, the monetary authority chooses a planned rate of inflation π_t^p. Actual inflation is then determined by the sum of π_t^p and a control error η_t;

$$\pi_t = \pi_t^p + \eta_t, \tag{3}$$

where η_t is iid normal with mean zero, standard deviation σ_η, and covariance $\sigma_{\varepsilon\eta}$ with ε_t. The monetary authority chooses π_t^p in order to minimize a loss function that penalizes variations of unemployment and inflation around target values $kU_t^n < U_t^n$ and zero:

$$L_t = (1/2)(U_t - kU_t^n)^2 + (b/2)\pi_t^2,$$

with $b > 0$.

Using (1) and (3), therefore, the monetary authority's problem can be written

$$\min_{\pi_t^p} E_{t-1}\{(1/2)[(1-k)U_t^n - \alpha(\pi_t^p - \pi_t^e + \eta_t)]^2 + (b/2)(\pi_t^p + \eta_t)^2\},$$

where $E_{t-1}(\cdot)$ denotes the expectation at the beginning of period t or, equivalently, at the end of period $t-1$. The first-order condition for this problem is

$$\alpha E_{t-1}[(1-k)U_t^n - \alpha(\pi_t^p - \pi_t^e + \eta_t)] = bE_{t-1}(\pi_t^p + \eta_t). \tag{4}$$

Private agents know the true structure of the economy and understand the monetary authority's time-consistency problem. In equilibrium, therefore, $\pi_t^e = \pi_t^p$. Using this condition as well as the fact that $E_{t-1}\eta_t = 0$, (4) simplifies to

$$\pi_t^p = \alpha A E_{t-1} U_t^n, \tag{5}$$

where

$$A = (1-k)/b > 0.$$

Equations (1) and (3), meanwhile, imply that

$$U_t = U_t^n - \alpha\eta_t. \tag{6}$$

Combining (2), (3), and (5) yields

$$\pi_t = \alpha A U_{t-1}^n + \alpha A \lambda \Delta U_{t-1}^n + \eta_t, \tag{7}$$

where $\Delta U_{t-1}^n = U_{t-1}^n - U_{t-2}^n$. Likewise, combining (2) and (6) yields

$$U_t = U_{t-1}^n + \lambda \Delta U_{t-1}^n + \varepsilon_t - \alpha\eta_t. \tag{8}$$

Together, (7) and (8) reveal that while both inflation π_t and unemployment U_t are nonstationary variables, inheriting a unit root from the underlying process for the natural rate U_t^n, the linear combination $\pi_t - \alpha A U_t$ is stationary:

$$\pi_t - \alpha A U_t = (1 + \alpha^2 A)\eta_t - \alpha A \varepsilon_t. \tag{9}$$

A relatively weak testable implication of the modified Barro-Gordon model, therefore, is that inflation and unemployment should be nonstationary, but cointegrated, variables.

Taking first differences of (6) yields

$$\Delta U_t = \Delta U_t^n - \alpha\eta_t + \alpha\eta_{t-1}. \tag{10}$$

Equations (9) and (10) indicate that for the purposes of estimation, the model can be written in state-space form (Hamilton, 1994, Ch.13) as

$$\xi_t = F\xi_{t-1} + Qv_t$$

and

$$y_t = H\xi_t,$$

where the 4×1 unobservable state vector ξ_t is given by

$$\xi_t = \begin{bmatrix} \Delta U_t^n \\ \varepsilon_t \\ \eta_t \\ \eta_{t-1} \end{bmatrix},$$

the 2×1 disturbance vector v_t is given by

$$v_t = \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix}$$

and has covariance matrix

$$Ev_tv_t' = \Sigma = \begin{bmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon\eta} \\ \sigma_{\varepsilon\eta} & \sigma_\eta^2 \end{bmatrix},$$

the 2×1 observation vector y_t is given by

$$y_t = \begin{bmatrix} \pi_t - \alpha AU_t \\ \Delta U_t \end{bmatrix},$$

the 4×4 matrix F is given by

$$F = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the 4×2 matrix Q is given by

$$Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the 2×4 matrix H is given by

$$H = \begin{bmatrix} 0 & -\alpha A & 1+\alpha^2 A & 0 \\ 1 & 0 & -\alpha & \alpha \end{bmatrix}.$$

Conditional on $\{y_{t-1}, y_{t-2}, ..., y_1\}$, y_t is normally distributed with mean $H\xi_{t|t-1}$ and variance $HP_{t|t-1}H'$, where $\{\xi_{t|t-1}\}_{t=1}^{T}$ and $\{P_{t|t-1}\}_{t=1}^{T}$ may be constructed recursively using the initial conditions

$$\xi_{1|0} = 0_{4\times 1}$$

and

$$vec(P_{1|0}) = [I_{16\times 16} - F \otimes F]^{-1} vec(Q\Sigma Q')$$

along with the updating equations

$$K_t = FP_{t|t-1}H'(HP_{t|t-1}H')^{-1},$$

$$\xi_{t+1|t} = F\xi_{t|t-1} + K_t(y_t - H\xi_{t|t-1}),$$

and

$$P_{t+1|t} = (F - K_t H)P_{t|t-1}(F' - H'K_t') + Q\Sigma Q'$$

for $t = 1, 2, ..., T-1$. Thus, the log likelihood function is

$$L = -T\ln(2\pi) + \sum_{t=1}^{T} L_t,$$

where

$$L_t = -\frac{1}{2}\ln[\det(HP_{t|t-1}H')] - \frac{1}{2}(y_t - H\xi_{t|t-1})'(HP_{t|t-1}H')^{-1}(y_t - H\xi_{t|t-1}).$$

The model may be estimated by choosing values for α, A, λ, σ_ε, σ_η, and $\sigma_{\varepsilon\eta}$ that maximize L.

Equation (9) shows that $\pi_t - \alpha A U_t$ is iid. Equation (10), meanwhile, can be rearranged to obtain

$$\Delta U_t^n = \Delta U_t + \alpha \eta_t - \alpha \eta_{t-1},$$

which, when substituted into (2), yields

$$\Delta U_t = \lambda \Delta U_{t-1} + \varepsilon_t - \alpha\lambda\eta_t + \alpha(1+\lambda)\eta_{t-1} - \alpha\lambda\eta_{t-2}.$$

Evidently, ΔU_t follows an ARMA(1,2). Hence, (9) and (10) can be viewed as a constrained vector ARMA(1,2) for a stationary linear combination of inflation and unemployment $\pi_t - \gamma U_t$ and the change in unemployment ΔU_t. An unconstrained model of this form is given by

$$\begin{aligned} \begin{bmatrix} \pi_t - \gamma U_t \\ \Delta U_t \end{bmatrix} &= \begin{bmatrix} \phi^{\pi\pi} & \phi^{\pi u} \\ \phi^{u\pi} & \phi^{uu} \end{bmatrix} \begin{bmatrix} \pi_{t-1} - \gamma U_{t-1} \\ \Delta U_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} \varepsilon_t^\pi \\ \varepsilon_t^u \end{bmatrix} + \begin{bmatrix} \theta_1^{\pi\pi} & \theta_1^{\pi u} \\ \theta_1^{u\pi} & \theta_1^{uu} \end{bmatrix} \begin{bmatrix} \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^u \end{bmatrix} + \begin{bmatrix} \theta_2^{\pi\pi} & \theta_2^{\pi u} \\ \theta_2^{u\pi} & \theta_2^{uu} \end{bmatrix} \begin{bmatrix} \varepsilon_{t-2}^\pi \\ \varepsilon_{t-2}^u \end{bmatrix}, \end{aligned}$$

where

$$E \begin{bmatrix} \varepsilon_t^\pi \\ \varepsilon_t^u \end{bmatrix} \begin{bmatrix} \varepsilon_t^\pi & \varepsilon_t^u \end{bmatrix} = \begin{bmatrix} \sigma_\pi^2 & \sigma_{\pi u} \\ \sigma_{\pi u} & \sigma_u^2 \end{bmatrix}.$$

The constrained model has 6 parameters, while the unconstrained model has 16 parameters. Thus, the Barro-Gordon model imposes 10 constraints on the time-series model.

The unconstrained model has the state-space representation

$$\xi_t = F\xi_{t-1} + Qv_t$$

$$y_t = H\xi_t,$$

where the 6×1 state vector ξ_t is given by

$$\xi_t = \begin{bmatrix} \pi_t - \gamma U_t \\ \Delta U_t \\ \varepsilon_t^\pi \\ \varepsilon_t^u \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^u \end{bmatrix},$$

the 2×1 disturbance vector v_t is given by

$$v_t = \begin{bmatrix} \varepsilon_t^{\pi} \\ \varepsilon_t^{u} \end{bmatrix}$$

and has covariance matrix

$$Ev_tv_t' = \Sigma = \begin{bmatrix} \sigma_\pi^2 & \sigma_{\pi u} \\ \sigma_{\pi u} & \sigma_u^2 \end{bmatrix},$$

the 2×1 observation vector y_t is given by

$$y_t = \begin{bmatrix} \pi_t - \gamma U_t \\ \Delta U_t \end{bmatrix},$$

the 6×6 matrix F is given by

$$F = \begin{bmatrix} \phi^{\pi\pi} & \phi^{\pi u} & \theta_1^{\pi\pi} & \theta_1^{\pi u} & \theta_2^{\pi\pi} & \theta_2^{\pi u} \\ \phi^{u\pi} & \phi^{uu} & \theta_1^{u\pi} & \theta_1^{uu} & \theta_2^{u\pi} & \theta_2^{uu} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

the 6×2 matrix Q is given by

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the 2×6 matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Conditional on $\{y_{t-1}, y_{t-2}, ..., y_1\}$, y_t is normally distributed with mean $H\xi_{t|t-1}$ and variance $HP_{t|t-1}H'$, where $\{\xi_{t|t-1}\}_{t=1}^T$ and $\{P_{t|t-1}\}_{t=1}^T$ may be constructed recursively using the initial conditions

$$\xi_{1|0} = 0_{6\times 1}$$

and

$$vec(P_{1|0}) = [I_{36\times 36} - F \otimes F]^{-1} vec(Q\Sigma Q')$$

along with the updating equations

$$K_t = FP_{t|t-1}H'(HP_{t|t-1}H')^{-1},$$

$$\xi_{t+1|t} = F\xi_{t|t-1} + K_t(y_t - H\xi_{t|t-1}),$$

and

$$P_{t+1|t} = (F - K_tH)P_{t|t-1}(F' - H'K_t^{'}) + Q\Sigma Q'$$

for $t = 1, 2, ..., T-1$. Thus, the log likelihood function is

$$L^u = -T\ln(2\pi) + \sum_{t=1}^{T} L_t^u,$$

where

$$L_t^u = -\frac{1}{2}\ln[\det(HP_{t|t-1}H')] - \frac{1}{2}(y_t - H\xi_{t|t-1})'(HP_{t|t-1}H')^{-1}(y_t - H\xi_{t|t-1}).$$

The model may be estimated by choosing values for the 16 parameters that maximize L^u. Under the null hypothesis that the constraints hold, the likelihood ratio statistic

$$2(L^u - L)$$

has a chi-square distribution with 10 degrees of freedom and can be used to test the constraints.

1.2. Testing for Unit Roots

Equations (7) and (8) show that according to the model, both the inflation rate and the unemployment rate ought to be nonstationary. The Phillips-Perron (1988) test described by Hamilton (1994, Ch.17) may be used to test for unit roots in these two variables. Since there is no obvious trend in either variable, the test procedure begins by estimating the regression equation

$$y_t = \alpha + \rho y_{t-1} + u_t$$

by OLS. Let $\hat{\rho}$ denote the OLS estimate of ρ, let $\hat{\sigma}_\rho$ denote the OLS standard error of ρ, and let $t = (\hat{\rho} - 1)/\hat{\sigma}_\rho$ denote the usual t-statistic.

In general, u_t will follow an MA(∞) process, so that the long-run variance of u_t, denoted λ^2, must be computed as suggested by Newey and West (1987). Let

$$\gamma_0 = T^{-1} \sum_{t=1}^{T} u_t^2$$

and, more generally, for $j = 1, 2, ..., q$, let

$$\gamma_j = T^{-1} \sum_{t=j+1}^{T} u_t u_{t-j}.$$

Then

$$\lambda^2 = \gamma_0 + 2 \sum_{j=1}^{q} [1 - j/(q+1)]\gamma_j.$$

Finally, let

$$s^2 = (T-2)^{-1} \sum_{t=1}^{T} u_t^2$$

denote the usual OLS estimate of the variance of u_t. Then the Phillips-Perron statistic

$$Z_t = (\gamma_0/\lambda^2)^{1/2} t - (1/2)[(\lambda^2 - \gamma_0)/\lambda](T\hat{\sigma}_\rho/s)$$

has critical values reported under the heading "case 2" in Hamilton's table B.6 (p.763).

In computing λ^2, q may be chosen as suggested by Andrews (1991). This procedure involves making the extra assumption that the process for u_t is well-approximated by an AR(1):

$$u_t = \pi u_{t-1} + \varepsilon_t.$$

With π estimated by OLS, the optimal choice for q is given by

$$q = 1.1447(\alpha T)^{1/3} - 1$$

where

$$\alpha = \frac{4\pi^2}{(1-\pi)^2(1+\pi)^2}.$$

The test statistics are tabulated below for the unemployment rate and the inflation rate.

	$\hat{\rho}$	t	q	Z_t
unemployment rate	0.9519	-1.7880	4	-2.5186
GDP deflator inflation	0.8788	-2.5362	0	-2.5362

The data are quarterly, and run from 1970:1 through 1997:2. Hence, $T = 110$. The unit root hypothesis cannot be rejected, even at the 90 percent confidence level, for the unemployment rate and the inflation rate. Notice that there is little evidence of serial correlation in the residuals from the inflation regression; hence, in this case, the Phillips-Perron test reduces to the tests originally developed by Dickey and Fuller (1979).

1.3. Testing for Cointegration

Equation (9) shows that according to the model, the inflation rate and the unemployment rate ought to be cointegrated. One approach to testing for cointegration is the Phillips-Ouliaris (1990) approach described by Hamilton (1994, Ch.19). This approach starts by estimating the regression equation

$$\pi_t = \alpha + \beta U_t + u_t$$

by OLS. Next, the residual u_t is regressed on its own lagged value:

$$u_t = \rho u_{t-1} + \varepsilon_t.$$

Let $\hat{\rho}$ denote the OLS estimate of ρ, let $\hat{\sigma}_\rho$ denote the OLS standard error of ρ, and let $t = (\hat{\rho} - 1)/\hat{\sigma}_\rho$ denote the usual t-statistic for the null hypothesis that $\rho = 1$.

If ε_t is serially correlated, its long-run variance, denoted λ^2, must be computed as suggested by Newey and West (1987). Let

$$\gamma_0 = (T-1)^{-1} \sum_{t=1}^{T} \varepsilon_t^2$$

and, more generally, for $j = 1, 2, ..., q$, let

$$\gamma_j = (T-1)^{-1} \sum_{t=j+1}^{T} \varepsilon_t \varepsilon_{t-j}.$$

Then

$$\lambda^2 = \gamma_0 + 2 \sum_{j=1}^{q} [1 - j/(q+1)] \gamma_j.$$

Finally, let

$$s^2 = (T-2)^{-1} \sum_{t=1}^{T} \varepsilon_t^2$$

denote the usual OLS estimate of the variance of ε_t. Then the

$$Z_t = (\gamma_0/\lambda^2)^{1/2}t - (1/2)[(\lambda^2 - \gamma_0)/\lambda][(T-1)\hat{\sigma}_\rho/s]$$

has critical values reported in Hamilton's table B.9 (p.766). Hamilton's "case 1" refers to the case in which the constant α is omitted from the initial cointegrating regression; "case 2" refers to the case in which the constant is included in the regression.

Again, the data are quarterly, and run from 1970:1 through 1997:2. The test results are tabulated below for the case suggested by the theory: without a constant.

$\hat{\beta}$	$\hat{\rho}$	t	q	Z_t
0.1791	0.8709	-2.7603	0	-2.7603

The null hypothesis of no cointegration can be rejected at the 95 percent confidence level. Hence, the test suggests that the data are consistent with this weak implication of the Barro-Gordon model. And there is no evidence of serial correlation in ε_t; Andrews' (1991) method dictates a choice of $q = 0$.

One potential weakness of the residual-based, Phillips-Ouliaris approach to testing for cointegration, discussed by Hamilton (1994, pp.589-590), is that the results in finite samples can depend on which variable, inflation or unemployment, is used as the dependent variable in the cointegrating regression. Here, however, (9) indicates that the cointegrating relationship is of the form $\pi_t - \gamma U_t$, making inflation the obvious choice of dependent variable.

Nevertheless, the robustness of the results obtained with the residual-based approach can be assessed by also testing for cointegration using the Johansen's (1988) maximum likelihood approach, as described by Hamilton (1994, Ch.20), which does not require a choice of normalization. The Johansen approach proceeds as follows.

Let y_t be the 2×1 vector

$$y_t = \begin{bmatrix} \pi_t \\ U_t \end{bmatrix}.$$

Next, suppose that y_t follows a VAR(p) in levels; this VAR can be written in the form

$$\Delta y_t = \xi_1 \Delta y_{t-1} + \xi_2 \Delta y_{t-2} + ... + \xi_{p-1} \Delta y_{t-p+1} + \xi_0 y_{t-1} + \varepsilon_t,$$

with $E\varepsilon_t\varepsilon_t' = \Omega$. Suppose, also, that π_t and U_t are both I(1), but a linear combination of the two variables is stationary. This implies that

$$\xi_0 = -BA',$$

where A and B are both 2×1 vectors. Under the assumption that the disturbances ε_t are Gaussian, the log likelihood function takes the usual form. Johansen's develops an algorithm for finding the maximum likelihood estimates without actually setting up and maximizing the likelihood function.

The first set is to estimate a VAR$(p-1)$ for Δy_t; this simply means estimating the system

$$\Delta y_t = \Pi_1 \Delta y_{t-1} + \Pi_2 \Delta y_{t-2} + ... + \Pi_{p-1} \Delta y_{t-p+1} + u_t$$

using equation-by-equation OLS. Then estimate a second set of auxiliary regressions of the form

$$y_{t-1} = \Theta_1 \Delta y_{t-1} + \Theta_2 \Delta y_{t-2} + ... + \Theta_{p-1} \Delta y_{t-p+1} + v_t.$$

Obviously, both u_t and v_t are 2×1.

Next, calculate the sample covariance matrices of u_t and v_t:

$$\Sigma_{uu} = (1/T) \sum_{t=1}^{T} u_t u_t',$$

$$\Sigma_{uv} = (1/T) \sum_{t=1}^{T} u_t v_t',$$

$$\Sigma_{vu} = \Sigma_{uv}',$$

and

$$\Sigma_{vv} = (1/T) \sum_{t=1}^{T} v_t v_t',$$

where each matrix is 2×2. From these, find the eigenvalues $\lambda_1 > \lambda_2$ of the 2×2 matrix

$$\Sigma_{vv}^{-1} \Sigma_{vu} \Sigma_{uu}^{-1} \Sigma_{uv}.$$

Let a_1 and a_2 be the associated eigenvectors. Johansen suggests normalizing these so that $a_i' \Sigma_{vv} a_i = 1$ for $i = 1, 2$; this can be done easily by letting $a_i = a_i / (a_i' \Sigma_{vv} a_i)^{1/2}$. The maximized value of the log likelihood function, attained subject to the constraint that there are h cointegrating relationships, is given by

$$L_h = -T \ln(2\pi) - T - (T/2) \log[\det(\Sigma_{uu})] - (T/2) \sum_{i=1}^{h} \ln(1 - \lambda_i).$$

The final step is to calculate the maximum likelihood estimates themselves. Under the assumption that there is one cointegrating vector:

$$A = a_1,$$

$$\xi_0 = \Sigma_{uv} A A',$$

$$\xi_i = \Pi_i - \xi_0 \Theta_i$$

for $i = 1, 2, ..., p-1$, and

$$\Omega = (1/T) \sum_{t=1}^{T} (u_t - \xi_0 v_t)(u_t - \xi_0 v_t)'.$$

The preceding analysis assumes that there are is no constant in the cointegrating regression, as implied by (9), and that there are no deterministic trends in the data; under these assumptions, no constant terms are included in the preliminary regressions.

To test the null hypothesis of no cointegrating relationship against the alternative of one cointegrating relationship, one can uses the likelihood ratio statistic $2(L_0 - L_1)$, which has the particularly simple form

$$2(L_1 - L_0) = -T \ln(1 - \lambda_1).$$

Critical values for this test statistic are reported under the heading "case 1" in Hamilton's table B.11. The number of random walks, which Hamilton denotes g, equals the number of variables n minus the number of cointegrating relationships h under the null hypothesis. Thus, in this case, $g = n - h = 2 - 0 = 2$. Since two cointegrating relationships would imply that both variables are stationary in levels, a test of this hypothesis seems redundant, given the results of the unit root tests.

The results from the Johansen approach are tabulated below.

λ_1	λ_2	A'	$2(L_1 - L_0)$
0.1189	0.0050	$\begin{bmatrix} 1.6985 & -0.3303 \end{bmatrix}$	13.6701

The null hypothesis of no cointegration can be rejected in favor of the hypothesis of one cointegrating vector at the 97.5 percent confidence level. Furthermore, renormalizing the estimated cointegrating relationship

$$1.6985 \pi_t = 0.3303 U_t$$

yields

$$\pi_t = 0.1945 U_t,$$

which is quite similar to the OLS estimate

$$\pi_t = 0.1791 U_t$$

obtained from the Phillips-Ouliaris approach. Regardless of which testing approach is used, therefore, the data appear to be consistent with the Barro-Gordon model's implication that inflation and unemployment are cointegrated.

1.4. Estimating the Model

Maximum likelihood estimates of the parameters of the constrained model are presented in the table below along with their standard errors, computed by taking square roots of the diagonal elements of the inverse of the information matrix. The estimates are obtained using the Kalman filter, as described above. The starting value for each parameter is also given in the table, although experiments with various alternative starting values yielded identical parameter estimates. In practice, the estimation procedure constrained the autoregressive parameter λ to lie between -1 and 1 by searching instead over values of ϕ and using the transformation

$$\lambda = \frac{\phi}{1 + |\phi|}.$$

In addition, to make sure that the estimate of covariance matrix Σ remained positive definite, the estimation procedure searched over values of the elements of the Cholesky decomposition Ω of Σ:

$$\Omega = \begin{bmatrix} \omega_\pi & 0 \\ \omega_{\pi u} & \omega_u \end{bmatrix},$$

where

$$\Sigma = \Omega\Omega'.$$

Both of these techniques are suggested by Hamilton (1994, pp.146-147).

Starting Value	Parameter	Estimate	Standard Error
0.5	α	0.1537	0.0641
0.5	A	1.1744	0.4894
0.5	λ	0.5505	0.0810
1	σ_ε	0.2905	0.0202
1	σ_η	0.6530	0.0446
0	$\sigma_{\varepsilon\eta}$	0.0725	0.0200

$$L \quad = \quad \text{maximized value of log likelihood} \; = -119.8246$$

The parameter estimates are all quite reasonable and the standard errors are small. The estimate $\alpha = 0.1537$ suggests that the Phillips curve is fairly steep: a one percentage point forecast error in inflation translates into only a 0.15 percentage point decline in the unemployment rate. The estimate of $A = (1-k)/b$ is greater than one; although k and b are not individually identified, the restriction $1 > k > 0$ implies that b must be less than one. Evidently, the Federal Reserve placed more weight on its goals for unemployment than its goals for inflation over the sample period. In addition, using (9), the estimated cointegrating vector for inflation and unemployment is given by

$$\pi_t = \alpha A U_t = 0.1805 U_t,$$

which is quite similar to the OLS estimate

$$\pi_t = 0.1791 U_t$$

obtained from the Phillips-Ouliaris approach. Finally, the estimate $\sigma_{\varepsilon\eta} = 0.0725 > 0$ indicates that unfavorable shocks to the natural rate of unemployment tended to coincide with unfavorable shocks to inflation; this finding is consistent with the idea that ε_t represents a real, or supply-side, shock.

1.5. Testing the Model

Maximum likelihood estimates of the parameters of the unconstrained model are presented in the table below along with their standard errors, computed by taking square roots of the diagonal elements of the inverse of the information matrix. The estimates are obtained using the Kalman filter, as described above. The starting values for each parameter, also reported in the table, were obtained by estimating a vector VAR(1) for the linear combination $\pi_t - 0.1791U_t$ of unemployment and inflation and the change in unemployment ΔU_t, where the estimate of the cointegrating vector is taken from Phillips-Ouliaris regression discussed above. To make sure that the estimate of the covariance matrix Σ remained positive definite, the estimation procedure searched over values of the elements of the Cholesky decomposition Σ, as in the constrained case and as suggested by Hamilton (1994, p.147). Unlike the constrained case, however, the ϕ parameters were not transformed during estimation.

Starting Value	Parameter	Estimate	Standard Error
0.18	γ	0.1978	0.0179
0.91	$\phi^{\pi\pi}$	1.1796	0.0898
-0.25	$\phi^{\pi u}$	-0.7315	0.2247
0.16	$\phi^{u\pi}$	0.2363	0.0818
0.54	ϕ^{uu}	0.4092	0.1748
0	$\theta_1^{\pi\pi}$	-0.5645	0.1341
0	$\theta_1^{\pi u}$	0.2067	0.2592
0	$\theta_1^{u\pi}$	-0.1125	0.1089
0	θ_1^{uu}	0.0870	0.1634
0	$\theta_2^{\pi\pi}$	-0.2152	0.1091
0	$\theta_2^{\pi u}$	0.6326	0.2152
0	$\theta_2^{u\pi}$	-0.0939	0.0800
0	θ_2^{uu}	0.1191	0.1382
$(0.094)^{1/2}$	σ_π	0.2711	0.0186
$(0.071)^{1/2}$	σ_u	0.2657	0.0182
-0.0096	$\sigma_{\pi u}$	-0.0144	0.0074

L^u = maximized value of log likelihood $= -21.7904$

The parameter estimates are all quite reasonable. Hamilton (1994, Ch.10, p.259) and Harvey (1981, Ch.2, p.51) state the condition that must hold if the ARMA(1,2) process is to be stationary; the roots of

$$|I\lambda - \Phi| = 0$$

must both lie inside the unit circle, where

$$\Phi = \begin{bmatrix} \phi^{\pi\pi} & \phi^{\pi u} \\ \phi^{u\pi} & \phi^{uu} \end{bmatrix}.$$

The roots of this equation coincide with the eigenvalues of Φ; hence, stationarity requires that both eigenvalues of Φ lie inside the unit circle. In fact, the eigenvalues of Φ are

$$\lambda_1 = 0.7944 + 0.1565i$$

and

$$\lambda_2 = 0.7944 - 0.1565i,$$

so that indeed

$$|\lambda_1| = |\lambda_2| = 0.8097 < 1,$$

indicating that the model is stationary.

Harvey (1981, Ch.2, p.51) states the condition that must hold if the model is to be invertible; the roots of

$$|I\lambda^2 + \Theta_1\lambda + \Theta_2| = 0$$

must all lie inside the unit circle, where

$$\Theta_1 = \begin{bmatrix} \theta_1^{\pi\pi} & \theta_1^{\pi u} \\ \theta_1^{u\pi} & \theta_1^{uu} \end{bmatrix}$$

and

$$\Theta_2 = \begin{bmatrix} \theta_2^{\pi\pi} & \theta_2^{\pi u} \\ \theta_2^{u\pi} & \theta_2^{uu} \end{bmatrix}.$$

The roots of this equation coincide with the eigenvalues of the matrix

$$\Theta = \begin{bmatrix} -\theta_1^{\pi\pi} & -\theta_1^{\pi u} & -\theta_2^{\pi\pi} & -\theta_2^{\pi u} \\ -\theta_1^{u\pi} & -\theta_1^{uu} & -\theta_2^{u\pi} & -\theta_2^{uu} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Hence, invertibility requires that all four eigenvalues of Θ lie inside the unit circle. In fact, the eigenvalues of Θ are

$$\lambda_1 = 0.4965 + 0.1148i,$$

$$\lambda_2 = 0.4965 - 0.1148i,$$

$$\lambda_3 = -0.2577 + 0.2524i,$$

and

$$\lambda_4 = -0.2577 - 0.2524i,$$

so that indeed

$$|\lambda_1| = |\lambda_2| = 0.5096 < 1$$

and

$$|\lambda_3| = |\lambda_4| = 0.3607 < 1,$$

indicating that the model is invertible as well.

The table below compares the estimates of the cointegrating vector $\pi_t - \gamma U_t$ obtained by each method.

Estimation Method	Estimate of γ, $\pi_t - \gamma U_t$
Phillips-Ouliaris	0.1791
Johansen	0.1945
Constrained Model	0.1805
Unconstrained Model	0.1978

The estimates are all quite similar.

The likelihood ratio statistic for the test of the Barro-Gordon model's restrictions is given by

$$2(L^u - L) = 2(119.8246 - 21.7904) = 196.0684.$$

The 99.9 percent critical value for a chi-square random variable with 10 degrees of freedom is 29.6 (Hamilton, table B.2, p.754). The model's implications are overwhelmingly rejected.

2. References

Andrews, Donald W.K. "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation." *Econometrica* 59 (May 1991): 817-858.

Barro, Robert J. and David B. Gordon. "A Positive Theory of Monetary Policy in a Natural Rate Model." *Journal of Political Economy* 91 (August 1983): 589-610.

Dickey, David A. and Wayne A. Fuller. "Distribution of the Estimators for Autoregressive Time Series With a Unit Root." *Journal of the American Statistical Association* 74 (June 1979): 427-431.

Hamilton, James D. *Time Series Analysis*. Princeton: Princeton University Press, 1994.

Harvey, A.C. *Time Series Models*. Oxford: Philip Allan Publishers Limited, 1981.

Johansen, Soren. "Statistical Analysis of Cointegration Vectors." *Journal of Economic Dynamics and Control* 12 (June/September 1988): 231-254.

Newey, Whitney K. and Kenneth D. West. "A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix." *Econometrica* 55 (May 1987): 703-725.

Phillips, Peter C.B. and Pierre Perron. "Testing for a Unit Root in Time Series Regression." *Biometrika* 75 (June 1988): 335-346.

Phillips, P.C.B. and S. Ouliaris. "Asymptotic Properties of Residual Based Tests for Cointegration." *Econometrica* 58 (January 1990): 165-193.