On the Fiscal Implications of Twin Crises
Technical Appendix

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April 2001

1. Key Elements of the Model

Government Transfers We assume that the economy is in initially in a steady state with a constant flow of lump-sum transfers from the government to households equal to v. Initially agents assume that this constant flow will persist indefinitely. At time 0 agents learn about a future increase in lump-sum transfers beginning at time $T' \gg 0$. Specifically we assume that

$$\begin{cases} v_t = v & \text{for } 0 \leq t < T', \\ v_t \geq v & \text{for } t \geq T'. \end{cases}$$

and

$$\int_0^\infty e^{-rt}(v_t - v)dt = \int_{T'}^\infty e^{-rt}(v_t - v)dt = \phi > 0. \tag{1.1}$$

Thus, ϕ denotes the present value of the increase in transfers.

Money Demand We assume that the demand for money takes the familiar Cagan form:

$$\ln(\frac{M_t}{P_t}) = \ln(\theta) + \ln(y_t) - \eta R_t, \tag{1.2}$$

where M_t is nominal money balances, P_t is the consumer price index, y_t is real GDP, and R_t is the nominal interest rate. Since our model is one in which agents have perfect foresight from period 0 forward, we have $R_t = r + \pi_t$ where $\pi_t \equiv d\ln(P_t)/dt = \dot{P}_t/P_t$.

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Tradables and Nontradables We allow for two types of goods: tradable goods whose wholesale price is given by P_t^T, and nontradable goods whose price is given by P_t^{NT}. We assume that selling a unit of the tradable good in the domestic market requires the use of δ units of the nontradable good.[1] Hence, the retail price of tradables is given by $P_t^T + \delta P_t^{NT}$.

The Consumer Price Index The CPI is defined as the geometric average of the price of nontradables and the retail price of tradables:

$$P_t \equiv (P_t^T + \delta P_t^{NT})^{\omega} (P_t^{NT})^{1-\omega}.$$

Purchasing Power Parity We assume that PPP holds for the wholesale price of tradable goods: $P_t^T = S_t P_t^{T*}$, where S_t is the nominal exchange rate expressed as units of local currency per unit of foreign currency, and P_t^{T*} is the foreign price of tradable goods. To simplify we normalize the foreign price of tradables to one, i.e. we let $P_t^{T*} = 1$. Hence PPP implies

$$P_t^T = S_t.$$

Local Price Stickiness Motivated by the data we assume that $P_t^{NT} = 1$ until some period $T_1 > 0$. At time T_1 the price of nontradables will start growing at the same rate as P_t.

The Time of a Speculative Attack We use the notation t^* to denote the time at which the speculative attack against the local currency takes place, and the fixed exchange rate regime is abandoned.

The Path of Output To model the effect on government finances of a possible recession in the wake of a speculative attack, we assume that the level of output is constant and equal to y until t^*. Between periods t^* and $t^* + \Delta$ we assume that output declines exponentially at the rate ρ. Then from $t^* + \Delta$ to $t^* + 2\Delta$ we assume that output increases exponentially at the rate ρ. From $t^* + 2\Delta$ forward, we assume that output is once again constant at the level y. To summarize

[1] We think of the nontradables as providing distribution services in the retail sector.

$$y_t = \begin{cases} y & \text{for } 0 \leq t < t^* \\ y e^{-\rho(t-t^*)} & \text{for } t^* \leq t \leq t^* + \Delta \\ y_{t^*+\Delta} e^{\rho[t-(t^*+\Delta)]} & \text{for } t^* + \Delta < t < t^* + 2\Delta \\ y & \text{for } t \geq t^* + 2\Delta. \end{cases} \tag{1.3}$$

Government Debt The government that has two types of debt. Dollar-denominated debt is denoted by b_t. This debt earns the interest rate r in dollar terms, or $\tilde{R}_t = r + \pi_t^T$, in local currency terms. Here $\pi_t^T \equiv \dot{P}_t^T / P_t$, denotes the inflation rate for the wholesale price of tradables.

In addition, we assume that prior to time 0 the government had issued D consols, yielding a constant coupon denominated in local currency. These nominal bonds were issued before agents learned about the increase in transfers. The coupon rate is equal to the nominal interest rate before the shock—since expected inflation was zero before the shock, this implies that the value of the bond is the same as its face value. To simplify our notation we assume that all debt issued after time 0 is dollar-denominated.

Government Spending In addition to the transfers, v_t, referred to above (which we assume are measured in units of the tradable good), we imagine that the government has additional spending commitments. We assume that the government purchases a constant quantity of tradable goods, g^T, at the price P_t^T. Furthermore, we assume that the government is commited to purchasing a constant quantity of nontradable goods, g^{NT}, at the price P_t^{NT}.[2] Finally, we assume that the government has some transfer commitments denominated in units of local currency, Z_t.[3] Total real spending by the government, exclusive of v_t, measured in units of the tradable good is $g^T + (P_t^{NT} g^{NT} + Z_t)/P_t^T$. For symmetry we will assume that Z_t is constant at the level Z until some time $T_2 \geq T_1$. After time T_2 it increases exponentially at the same rate as P_t.

Taxes We assume that the government raises tax revenue at a constant rate τ, so that revenue in units of local currency is given by $P_t^T \tau y_t$.

[2] One example of such a commitment might be the government wage bill.
[3] One example might be social security payments.

The Government Budget Constraint The government's flow budget constraint for $t \geq 0$ is

$$\dot{B}_t = \tilde{R}_t B_t + rD + P_t^T (g^T + v_t - \tau y_t) + P_t^{NT} g^{NT} + Z_t - \dot{M}_t.$$

where $B_t = P_t^T b_t$ is the local-currency value of the dollar-denominated bonds. This implies that

$$\dot{b}_t = r b_t + \frac{rD}{P_t^T} + g^T + v_t - \tau y_t + \frac{P_t^{NT} g^{NT} + Z_t}{P_t^T} - \dot{m}_t^T - \pi_t^T m_t^T \tag{1.4}$$

where $m_t^T = M_t / P_t^T$ is real balances in units of the tradable good.[4]

At some points in time, $t \in I$, the money supply may jump discretely, although the price level will not. At such instants in time, the government's budget constraint is $\Delta B_t = -\Delta M_t$, where ΔX_t is the magnitude of the jump in variable X_t at time t. It follows that $\Delta b_t = -\Delta m_t^T$.

Initially Sustainable Government Finances We assume that before the shock at time 0, a fixed exchange rate regime in which $S_t = S$ for all t would have been sustainable. By this we mean that with $S_t = S$, $v_t = v$, $Z_t = Z$, $y_t = y$ for all t, the paths of P_t^T, P_t^{NT}, P_t, M_t and b_t would all be constant with $P_t^T = S$, $P_t^{NT} = 1$, $P_t = P = (S + \delta)^\omega$, $M_t = M = (\theta e^{-\eta r}) P y$ and $b_t = b_0$, such that

$$r(b_0 + D/S) = \tau y - g^T - v - \frac{g^{NT} + Z}{S}. \tag{1.5}$$

The Threshold Rule Under the fixed exchange rate regime, the government must accomodate money demand. We assume that the government immediately abandons the fixed exchange rate regime if money demand falls to the level $e^{-\chi} M$ for $\chi > 0$.

New Monetary Policy Given the increase in transfers announced at time 0, we assume that the government implements a new monetary policy. At some period $T > 0$ we assume that the government increases the money supply to the level $M_T = e^\gamma M$ for some $\gamma > 0$.

[4] To see that this condition follows, notice that the budget constraint implies

$$\frac{\dot{B}_t}{P_t^T} = \tilde{R} b_t + \frac{rD}{P_t^T} + g^T + v_t - \tau y_t + \frac{P_t^{NT} g^{NT} + Z_t}{P_t^T} - \frac{\dot{M}_t}{P_t^T}.$$

Since $B_t = b_t P_t^T$ we have $\dot{B}_t = \dot{b}_t P_t^T + b_t \dot{P}_t^T$ and $\dot{B}_t / P_t^T = \dot{b}_t + \pi_t^T b_t$. Similarly, $\dot{M}_t / P_t^T = \dot{m}_t + \pi_t^T m_t$. Since $\tilde{R} = r + \pi_t^T$, the equation in the text follows.

Furthermore, we assume that from date T forward, the money growth rate is μ, so that $M_t = M_T e^{\mu(t-T)}$ for $t \geq T$.

What about the period prior to T? Initially the government keeps the exchange rate fixed. Suppose the exchange rate stays fixed up to time t^*. Then for $0 \leq t < t^*$, $M_t = M$. As we saw above, the government would abandon the fixed exchange rate regime at time t^* if money demand fell to the level $e^{-\chi}M$. We assume that for $t^* \leq t < T$ the money supply remains at this level.

Timing We assume that $T_2 \geq T_1 \geq T$ and that $T' \geq T$.

2. Solving the Model

To solve the model we calculate t^* given the various parameters describing the economy and government policy. It turns out that given all the other parameters, the parameters of monetary policy cannot be set arbitrarily. For example, given arbitrary values of T and γ, the value of μ is restricted by the condition that the government must satisfy its lifetime budget constraint. Hence, in this case, we must solve simultaneously for μ and t^*. Alternatively we can set T and μ and solve for γ and t^*.

Fortunately we can obtain a closed-form solution for t^* in terms of an arbitrary choice of μ, γ and T (and other parameters). This considerably simplifies the process of finding a (μ, γ, T) triple such that the government's lifetime budget constraint is satisfied.

In this section, we first describe how we solve for t^*. We then describe our numerical algorithm for obtaining, say, the value of μ, given arbitrary values of all the other parameters.

2.1. Solving for t^*

The key equation in determining the time of the speculative attack is the money demand function, (1.2), which can be used to solve for the price level for any time t at which exchange rate is floating. Notice that (1.2) can be rewritten as a differential equation in $p_t \equiv \ln P_t$:

$$p_t = \eta r - \ln\theta + \ln(M_t/y_t) + \eta \dot{p}_t.$$

It follows that

$$\ln P_t = \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln(M_s/y_s)ds. \tag{2.1}$$

is the solution.[5]

By definition, the currency is floated at time t^*, so that

$$\ln P_{t^*} = \eta r - \ln\theta + \frac{1}{\eta}\int_{t^*}^{\infty} \ln(M_s/y_s)e^{-(s-t^*)/\eta}ds. \tag{2.2}$$

The instant before the currency is floated, we know, from above, that money demand is given by

$$\ln M - \ln P = \ln\theta + \ln y - \eta r. \tag{2.3}$$

Furthermore, we know that the absence of arbitrage opportunities in currency trading requires that $S_{t^*} = S$. This combined with the fact that $P_{t^*}^{NT} = 1$ implies that $P_{t^*} = P = (S+\delta)^{\omega}$. Hence $\ln P_{t^*} = \ln P = \eta r - \ln\theta + \ln M - \ln y$. Combining this result with (**??**), we get

$$\begin{aligned} \ln M - \ln y &= \frac{1}{\eta}\int_{t^*}^{\infty} \ln(M_s/y_s)e^{-(s-t^*)/\eta}ds \\ &= \frac{1}{\eta}\int_{t^*}^{\infty} \ln M_s e^{-(s-t^*)/\eta}ds - \frac{1}{\eta}\int_{t^*}^{\infty} \ln y_s e^{-(s-t^*)/\eta}ds. \end{aligned} \tag{2.4}$$

Our next step is to evaluate the two integrals on the right-hand side of (2.4). The first of these integrals depends on the path of the money supply after the speculative attack. The monetary policy rule we described above implies that $M_t = e^{-\chi}M$ for $t^* \le t < T$, and $M_t = e^{\gamma+\mu(t-T)}M$ for $t \ge T$.[6] Hence, the first term on the right hand side of (2.4) is

$$\frac{1}{\eta}\int_{t^*}^{\infty} \ln M_s e^{-(s-t^*)/\eta}ds = \frac{1}{\eta}\int_{t^*}^{T} \ln M_s e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{T}^{\infty} \ln M_s e^{-(s-t^*)/\eta}ds$$

In the appendix we show that this can be rewritten as

$$\frac{1}{\eta}\int_{t^*}^{\infty} \ln M_s e^{-(s-t^*)/\eta}ds = \ln M - \chi\left[1 - e^{-(T-t^*)/\eta}\right] + (\gamma+\mu\eta)e^{-(T-t^*)/\eta}. \tag{2.5}$$

[5] To verify that this is indeed the solution differentiate both sides with respect to t to obtain

$$\dot{p}_t = -\frac{1}{\eta}\ln(M_t/y_t) + \frac{1}{\eta^2}\int_t^{\infty} e^{-(s-t)/\eta}\ln(M_s/y_s)ds$$

which implies

$$\begin{aligned} \eta\dot{p}_t &= -\ln(M_t/y_t) + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln(M_s/y_s)ds \\ &= -\ln(M_t/y_t) + p_t - a. \end{aligned}$$

[6] Implicit in this description is the assumption that a solution for t^* such that $t^* < T$ exists. We will see that this assumption is valid.

The second integral on the right-hand side of (2.4) can be evaluated using (1.3) as follows:

$$\begin{aligned}\frac{1}{\eta}\int_{t^*}^{\infty}\ln y_s e^{-(s-t^*)/\eta}ds &= \frac{1}{\eta}\int_{t^*}^{t^*+\Delta}\ln y_s e^{-(s-t^*)/\eta}ds + \\ &\quad \frac{1}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta}\ln y_s e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{t^*+2\Delta}^{\infty}\ln y_s ds\end{aligned}$$

In the appendix we show that this can be rewritten as

$$\frac{1}{\eta}\int_{t^*}^{\infty}\ln y_s e^{-(s-t^*)/\eta}ds = \ln y - \rho\eta\left(1-e^{-\Delta/\eta}\right)^2. \tag{2.6}$$

Subsitituting (2.5) and (2.6) into (2.4) we get

$$\chi - \rho\eta\left(1-e^{-\Delta/\eta}\right)^2 = (\chi+\gamma+\mu\eta)e^{-(T-t^*)/\eta}.$$

This equation can be solved for t^*. The time of the speculative attack is given by

$$t^* = T - \eta\ln\left[\frac{\chi+\gamma+\mu\eta}{\chi-\rho\eta\left(1-e^{-\Delta/\eta}\right)^2}\right] \tag{2.7}$$

Since the numerator is positive, we must assume that

$$\chi > \rho\eta\left(1-e^{-\Delta/\eta}\right)^2$$

in order to obtain a solution for the time of the attack. Second, notice that if a solution exists $t^* < T$ since the numerator is larger than the denominator. Finally, note that if the solution implied by (2.7) is less than 0, the attack happens immediately, i.e. $t^* = 0$.

In the case where there is no post-attack recession, the second term in the denominator is zero. That is

$$t^* = T - \eta\ln\left[\frac{\chi+\gamma+\mu\eta}{\chi}\right].$$

This means that the attack happens sooner if there is a post-attack recession than it would if output was constant. The reduction in output is a negative money demand shock. This means that if we keep the path of money constant, there is more inflation than in the model with constant output. These higher rates of inflation translate into a quicker attack.

This completes our determination of the time of the attack. We turn next to our analysis of the government's lifetime budget constraint.

2.2. The Government Budget Constraint

Above, we noted that the government's flow budget constraint is given by (1.4). We also noted that if we allow for jumps in real debt at specific points in time, these will satisfy: $\Delta b_t = -\Delta m_t^T$. Given our description of monetary policy, we know that the dates at which such jumps occur are t^* and T. Using this fact and (1.4), in the appendix we show that the government faces the following intertemporal budget constraint:

$$\begin{aligned} b_0 &= -\int_0^\infty \left[\frac{rD}{P_t^T} + g^T + v_t - \tau y_t + \frac{P_t^{NT} g^{NT} + Z_t}{P_t^T} - \dot{m}_t^T - \pi_t^T m_t^T\right] e^{-rt} dt + \\ & e^{-rt^*}\Delta m_{t^*}^T + e^{-rT}\Delta m_T^T. \end{aligned} \tag{2.8}$$

The last two parts of this expression reflect the jumps in real balances (and government debt) occurring at times t^* and T.

In the appendix we show that (2.8) can be rewritten as

$$\begin{aligned} \phi &= \int_{t^*}^\infty (\dot{m}_t^T + \pi_t^T m_t^T) e^{-rt} dt + e^{-rt^*}\Delta m_{t^*}^T + e^{-rT}\Delta m_T^T - \\ & \tau \int_{t^*}^{t^*+2\Delta} (y - y_t) e^{-rt} dt + \int_{t^*}^\infty \left(\frac{g^{NT}}{S} - \frac{g^{NT} P_t^{NT}}{S_t}\right) e^{-rt} dt + \\ & \int_{t^*}^\infty \left(\frac{Z}{S} - \frac{Z_t}{S_t}\right) e^{-rt} dt + \int_{t^*}^\infty \left(\frac{rD}{S} - \frac{rD}{S_t}\right) e^{-rt} dt. \end{aligned} \tag{2.9}$$

The left-hand side of (2.9) is the present value of the increase in lump-sum transfers (of tradables) due to the shock at time 0. The right-hand side of (2.9) represents the different sources of revenue generated or lost during and after the speculative attack at time t^*:

(i) the first three terms on the right hand side, $\int_{t^*}^\infty (\dot{m}_t^T + \pi_t^T m_t^T) e^{-rt} dt + e^{-rt^*}\Delta m_{t^*}^T + e^{-rT}\Delta m_T^T$, represent the present value of seignorage,

(ii) the fourth terms on the right hand side, $-\tau \int_{t^*}^{t^*+2\Delta} (y - y_t) e^{-rt} dt$, is present value of tax revenues lost as a result of the post-crisis recession,

(iii) the fifth term, $g^{NT} \int_{t^*}^\infty (1/S - P_t^{NT}/S_t) e^{-rt} dt$, represents the reduced cost of the government's spending on nontradables,

(iv) the sixth term, $\int_{t^*}^\infty (Z/S - Z_t/S_t) e^{-rt} dt$, represents the reduced cost of the government's local currency transfer commitments, and

(v) the seventh term, $rD \int_{t^*}^\infty (1/S - 1/S_t) e^{-rt} dt$, represents the devaluation of the government's nominal debt obligations acquired before the shock at time 0.

In the appendix we show that parts of (2.9) can be obtained analytically, while others must be calculated numerically. We turn next to a description of our algorithm for solving the model.

3. Algorithm

We calibrate many of the parameters describing our economy. The full list of parameters that we calibrate is η, χ, Δ, ϕ, . Let the vector containing all these parameters be Ψ. We calibrate the parameters in Ψ with reference to empirical evidence for each of the countries we examine. The remaining parameters are those describing the post-crisis monetary policy: T, γ and μ. As we stated before, if the expression for t^*, (2.7), is substituted into (2.9), the resulting equation is of the form $f(\Psi, T, \gamma, \mu) = 0$. Hence, given calibrated values of Ψ, T and γ, (2.9) can be solved for μ. However, this must be done numerically. Our algorithm for finding the equilibrium value of μ can be described as follows:

1. Fix Ψ, T and γ and guess a value for μ,
2. Use (2.7) to compute t^*,
3. Subsitute t^* into (2.9). If the two sides of (2.9) are approximately equal—within some small tolerance—μ is the solution and the algorithm stops. If they are not, the algorithm iterates by repeating step 1.

Once the algorithm has converged, we have the equilibrium values of the time of the attack, t^*, and the new steady state money growth rate, μ.

4. Appendix

4.1. Some Useful Formulas

Here we derive expressions for $\int_a^b e^{-(s-t)/\eta} ds$ and $\int_a^b s e^{-(s-t)/\eta} ds$. First, we note that for any function $e^{\psi x}$

$$\int e^{\psi x} dx = C + e^{\psi x}/\psi \tag{4.1}$$

and

$$\int x e^{\psi x} dx = C + (x - 1/\psi)\left(e^{\psi x}/\psi\right) \tag{4.2}$$

where in each case C is some arbitrary constant of integration. It is straightforward to verify these solutions. Notice that the derivative of the right-hand-side of (4.1) with respect to x is $e^{\psi x}$, while the derivative of the right-hand-side of (4.2) with respect to x is $xe^{\psi x}$.

Hence we can write

$$\begin{aligned}\int_a^b e^{-(s-t)/\eta}ds &= e^{t/\eta}\int_a^b e^{-s/\eta}ds \\ &= e^{t/\eta}(-\eta e^{-s/\eta})|_a^b \\ &= -\eta e^{t/\eta}(e^{-b/\eta}-e^{-a/\eta}) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_a^b se^{-(s-t)/\eta}ds &= e^{t/\eta}\int_a^b se^{-s/\eta}ds \\ &= e^{t/\eta}[(s+\eta)(-\eta e^{-s/\eta})]|_a^b \\ &= -\eta e^{t/\eta}[(b+\eta)e^{-b/\eta}-(a+\eta)e^{-a/\eta}]. \end{aligned} \tag{4.4}$$

4.2. Solving for t^*

In the section on solving for t^* we need simplified expressions for the following integrals: $\eta^{-1}\int_{t^*}^{\infty}\ln M_s e^{-(s-t^*)/\eta}ds$ and $\eta^{-1}\int_{t^*}^{\infty}\ln y_s e^{-(s-t^*)/\eta}ds$. We have

$$\frac{1}{\eta}\int_{t^*}^{\infty}\ln M_s e^{-(s-t^*)/\eta}ds = \frac{1}{\eta}\int_{t^*}^{T}\ln M_s e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{T}^{\infty}\ln M_s e^{-(s-t^*)/\eta}ds.$$

Since $M_t = Me^{-\chi}$ for $t^* \leq t < T$ and $M_t = Me^{\gamma+\mu(t-T)}$ for $t \geq T$ this becomes

$$\begin{aligned} &= \frac{1}{\eta}\int_{t^*}^{T}\ln(e^{-\chi}M)e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{T}^{\infty}\ln[e^{\gamma+\mu(s-T)}M]e^{-(s-t^*)/\eta}ds \\ &= \frac{\ln M-\chi}{\eta}\int_{t^*}^{T}e^{-(s-t^*)/\eta}ds + \frac{\ln M+\gamma-\mu T}{\eta}\int_{T}^{\infty}e^{-(s-t^*)/\eta}ds + \frac{\mu}{\eta}\int_{T}^{\infty}se^{-(s-t^*)/\eta}ds \\ &= (\ln M-\chi)\left[1-e^{-(T-t^*)/\eta}\right] + (\ln M+\gamma-\mu T)e^{-(T-t^*)/\eta} + \mu e^{-(T-t^*)/\eta}(T+\eta) \end{aligned}$$

In the last line we have used (4.3) and (4.4) derived above. Simplifying our expression further, we have

$$\frac{1}{\eta}\int_{t^*}^{\infty}\ln M_s e^{-(s-t^*)/\eta}ds = \ln M - \chi\left[1-e^{-(T-t^*)/\eta}\right] + (\gamma+\mu\eta)e^{-(T-t^*)/\eta}.$$

We also have:

$$\begin{aligned}\frac{1}{\eta}\int_{t^*}^{\infty}\ln y_s e^{-(s-t^*)/\eta}ds &= \frac{1}{\eta}\int_{t^*}^{t^*+\Delta}\ln y_s e^{-(s-t^*)/\eta}ds + \\ &\quad \frac{1}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta}\ln y_s e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{t^*+2\Delta}^{\infty}\ln y_s ds\end{aligned}$$

Using (1.3) this becomes

$$
\begin{aligned}
&= \frac{1}{\eta}\int_{t^*}^{t^*+\Delta} \ln[ye^{-\rho(s-t^*)}]e^{-(s-t^*)/\eta}ds + \frac{1}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} \ln[y_{t^*+\Delta}e^{\rho[s-(t^*+\Delta)]}]e^{-(s-t^*)/\eta}ds + \\
&\quad \frac{1}{\eta}\int_{t^*+2\Delta}^{\infty} \ln ye^{-(s-t^*)/\eta}ds \\
&= \frac{\ln y + \rho t^*}{\eta}\int_{t^*}^{t^*+\Delta} e^{-(s-t^*)/\eta}ds - \frac{\rho}{\eta}\int_{t^*}^{t^*+\Delta} se^{-(s-t^*)/\eta}ds + \\
&\quad \frac{\ln y - \rho(t^*+2\Delta)}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(s-t^*)/\eta}ds + \frac{\rho}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} se^{-(s-t^*)/\eta}ds + \\
&\quad \frac{\ln y}{\eta}\int_{t^*+2\Delta}^{\infty} e^{-(s-t^*)/\eta}ds \\
&= \frac{\ln y}{\eta}\int_{t^*}^{\infty} e^{-(s-t^*)/\eta}ds + \frac{\rho t^*}{\eta}\int_{t^*}^{t^*+\Delta} e^{-(s-t^*)/\eta}ds - \frac{\rho(t^*+2\Delta)}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(s-t^*)/\eta}ds - \\
&\quad \frac{\rho}{\eta}\int_{t^*}^{t^*+\Delta} se^{-(s-t^*)/\eta}ds + \frac{\rho}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} se^{-(s-t^*)/\eta}ds.
\end{aligned}
$$

Once more, we can use (4.3) and (4.4) to show that this means

$$
\begin{aligned}
\frac{1}{\eta}\int_{t^*}^{\infty} \ln y_s e^{-(s-t^*)/\eta}ds &= \ln y - \rho t^*[e^{-\Delta/\eta} - 1] + \rho(t^*+2\Delta)\left\{[e^{-2\Delta/\eta} - e^{-\Delta/\eta}]\right\} \\
&\quad -\frac{\rho}{\eta}\left\{-\eta[(t^*+\Delta+\eta)e^{-\Delta/\eta} - (t^*+\eta)]\right\} \\
&\quad +\frac{\rho}{\eta}\left\{-\eta[(t^*+2\Delta+\eta)e^{-2\Delta/\eta} - (t^*+\Delta+\eta)e^{-\Delta/\eta}]\right\} \\
&= \ln y - \rho\eta\left(1 - e^{-\Delta/\eta}\right)^2.
\end{aligned}
$$

4.3. The Government Budget Constraint

We consider two cases: $t^* = 0$ and $t^* > 0$.

4.3.1. When $t^* > 0$

Notice that (1.4) can be rewritten as $\dot{b}_t = rb_t + x_t$ where

$$
x_t = \frac{rD}{P_t^T} + g^T + v_t - \tau y_t + \frac{P_t^{NT}g^{NT} + Z_t}{P_t^T} - \dot{m}_t^T - \pi_t^T m_t^T.
$$

Multiplying through by e^{-rt} we have $\dot{b}_t e^{-rt} = rb_t e^{-rt} + x_t e^{-rt}$. Notice that this means

$$
\int_a^c \dot{b}_t e^{-rt}dt = \int_a^c rb_t e^{-rt}dt + \int_a^c x_t e^{-rt}dt. \tag{4.5}
$$

The standard rules of integration by parts imply that

$$\int_a^c \dot{b}_t e^{-rt} dt = e^{-rt} b_t|_a^c + r \int_a^c b_t e^{-rt} dt. \tag{4.6}$$

Combining (4.5) and (4.6) we have

$$e^{-rt} b_t|_a^c = \int_a^c x_t e^{-rt} dt. \tag{4.7}$$

If we let $a = T$ and $c = \infty$ we have

$$\lim_{t \to \infty} e^{-rt} b_t - e^{-rT} b_T = \int_T^\infty x_t e^{-rt} dt.$$

If we impose the no-Ponzi scheme condition, $\lim_{t \to \infty} e^{-rt} b_t = 0$ we obtain

$$e^{-rT} b_T = -\int_T^\infty x_t e^{-rt} dt. \tag{4.8}$$

Notice that b_T is the level of debt immediately after the jump in the money supply at time t. That is $b_T = \lim_{t \downarrow T} b_t$. There is a jump in real money balances at time T which we denote Δm_T^T. Hence if we define $b_{\underline{T}} \equiv \lim_{t \uparrow T} b_t$ we know that $b_T - b_{\underline{T}} = -\Delta m_T^T$ or

$$b_{\underline{T}} = b_T + \Delta m_T^T. \tag{4.9}$$

Once again we use (4.7), this time with $a = t^*$ and $c = T$ to write

$$e^{-rT} b_{\underline{T}} - e^{-rt^*} b_{t^*} = \int_{t^*}^T x_t e^{-rt} dt.$$

Using (4.8) and (4.9) this implies

$$\begin{aligned} e^{-rt^*} b_{t^*} &= e^{-rT}(b_T + \Delta m_T^T) - \int_{t^*}^T x_t e^{-rt} dt \\ &= -\int_{t^*}^\infty x_t e^{-rt} dt + e^{-rT} \Delta m_T^T. \end{aligned} \tag{4.10}$$

Notice that b_{t^*} is the level of debt immediately after the jump in the money supply at time t^*. That is $b_{t^*} = \lim_{t \downarrow t^*} b_t$. There is a jump in real money balances at time t^* which we denote $\Delta m_{t^*}^T$. Hence if we define $b_{\underline{t^*}} \equiv \lim_{t \uparrow t^*} b_t$ we know that $b_{t^*} - b_{\underline{t^*}} = -\Delta m_{t^*}^T$ or

$$b_{\underline{t^*}} = b_{t^*} + \Delta m_{t^*}^T. \tag{4.11}$$

Finally we use (4.7) again, this time with $a = 0$ and $c = t^*$ to write

$$e^{-rt^*} b_{\underline{t^*}} - b_0 = \int_0^{t^*} x_t e^{-rt} dt.$$

Using (4.10) and (4.11) this implies

$$\begin{aligned} b_0 &= e^{-rt^*}(b_{t^*} + \Delta m^T_{t^*}) - \int_0^{t^*} x_t e^{-rt} dt \\ &= -\int_0^{\infty} x_t e^{-rt} dt + e^{-rt^*}\Delta m^T_{t^*} + e^{-rT}\Delta m^T_T. \end{aligned} \tag{4.12}$$

We now rewrite (4.12) as

$$b_0 = -\int_0^{\infty} x e^{-rt} dt - \int_0^{\infty} (x_t - x) e^{-rt} dt + e^{-rt^*}\Delta m^T_{t^*} + e^{-rT}\Delta m^T_T \tag{4.13}$$

where x is the value that x_t would take on under the sustainable fixed exchange rate regime. I.e.

$$x = g^T + v - \tau y + \frac{g^{NT} + Z + rD}{S}.$$

Notice that (1.5) implies that $rb_0 = -x$. Hence (4.13) can be rewritten as

$$\int_0^{\infty} (x_t - x) e^{-rt} dt = e^{-rt^*}\Delta m^T_{t^*} + e^{-rT}\Delta m^T_T \tag{4.14}$$

Subsitituting in the definitions of x_t and x, and noting that $P^T_t = S_t$ we get

$$\int_0^{\infty} \left[v_t - v - \tau(y_t - y) + \frac{P^{NT}_t g^{NT} + Z_t + rD}{S_t} - \frac{g^{NT} + Z + rD}{S} \right] e^{-rt} dt =$$

$$\int_0^{\infty} (\dot{m}^T_t + \pi^T_t m^T_t) e^{-rt} dt + e^{-rt^*}\Delta m^T_{t^*} + e^{-rT}\Delta m^T_T. \tag{4.15}$$

We can rearrange this as

$$\begin{aligned} \int_0^{\infty} (v_t - v) e^{-rt} dt &= \int_0^{\infty} (\dot{m}^T_t + \pi^T_t m^T_t) e^{-rt} dt + e^{-rt^*}\Delta m^T_{t^*} + e^{-rT}\Delta m^T_T + \\ & \int_0^{\infty} \tau(y_t - y) e^{-rt} dt - \int_0^{\infty} \left(\frac{P^{NT}_t g^{NT}}{S_t} - \frac{g^{NT}}{S} \right) e^{-rt} dt - \\ & \int_0^{\infty} \left(\frac{Z_t}{S_t} - \frac{Z}{S} \right) e^{-rt} dt - \int_0^{\infty} \left(\frac{rD}{S_t} - \frac{rD}{S} \right) e^{-rt} dt \end{aligned} \tag{4.16}$$

The next step is to simplify (4.16). First, we note that $\int_0^{\infty} (v_t - v) e^{-rt} dt = \phi$. Then we simplify terms on the right hand side.

First term.—It requires us to evaluate $\int_0^{\infty} (\dot{m}^T_t + \pi^T_t m^T_t) e^{-rt} dt$. Recall that for $0 \le t < t^*$ we have $M_t = M$ and $S_t = S$, hence $\dot{m}^T_t = \pi^T_t = 0$. For $t \ge T_1$, (1.2) implies that $m^T_t = m^T_{T_1}$ and we know that $\pi^T_t = \mu$. Hence we have

$$\int_0^{\infty} (\dot{m}^T_t + \pi^T_t m^T_t) e^{-rt} dt = \int_{t^*}^{T_1} (\dot{m}^T_t + \pi^T_t m^T_t) e^{-rt} dt + \frac{\mu m^T_{T_1}}{r} e^{-rT_1}.$$

At this point it is convenient to note that

$$\dot{m}_t^T + \pi_t^T m_t^T = \dot{M}_t/S_t - (M_t/S_t^2)\dot{S}_t + \pi_t^T m_t^T = \dot{M}_t/S_t.$$

For $t^* \leq t < T$, $M_t = Me^{-\chi}$ so that $\dot{M}_t = 0$. Also, for $t \geq T$, $\dot{M}_t = \mu M_t$, so that

$$\int_0^\infty (\dot{m}_t^T + \pi_t^T m_t^T)e^{-rt}dt = \int_T^{T_1} \mu m_t^T e^{-rt}dt + \frac{\mu m_{T_1}^T}{r}e^{-rT_1}.$$

For $T \leq t < T_1$ we know that $M_t = M_T e^{\mu(t-T)}$, so that

$$\int_T^{T_1} \mu m_t^T e^{-rt}dt = \mu M_T e^{-\mu T}\int_T^{T_1} \frac{1}{S_t}e^{(\mu-r)t}dt$$

implying

$$\int_0^\infty (\dot{m}_t^T + \pi_t^T m_t^T)e^{-rt}dt = \mu M_T e^{-\mu T}\int_T^{T_1} \frac{1}{S_t}e^{(\mu-r)t}dt + \frac{\mu m_{T_1}^T}{r}e^{-rT_1}.$$

Fourth term.—Here, note that

$$\begin{aligned}\int_0^\infty (y_t - y)e^{-rt}dt &= y\int_{t^*}^{t^*+\Delta}(e^{-\rho(t-t^*)}-1)e^{-rt}dt + y\int_{t^*+\Delta}^{t^*+2\Delta}(e^{\rho(t-t^*)-2\rho\Delta}-1)e^{-rt}dt \\ &= ye^{\rho t^*}\int_{t^*}^{t^*+\Delta} e^{-(r+\rho)t}dt + ye^{-(\rho t^*+2\rho\Delta)}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(r-\rho)t}dt - y\int_{t^*}^{t^*+2\Delta} e^{-rt}dt\end{aligned}$$

Using (4.3) we have

$$\begin{aligned}ye^{\rho t^*}\int_{t^*}^{t^*+\Delta} e^{-(r+\rho)t}dt &= -\frac{ye^{\rho t^*}}{r+\rho}\left[e^{-(r+\rho)(t^*+\Delta)} - e^{-(r+\rho)t^*}\right] \\ &= ye^{-rt^*}\left[\frac{1-e^{-(r+\rho)\Delta}}{r+\rho}\right],\end{aligned}$$

$$\begin{aligned}ye^{-(\rho t^*+2\rho\Delta)}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(r-\rho)t}dt &= -\frac{ye^{-(\rho t^*+2\rho\Delta)}}{r-\rho}\left[e^{-(r-\rho)(t^*+2\Delta)} - e^{-(r-\rho)(t^*+\Delta)}\right] \\ &= ye^{-rt^*}\left[\frac{e^{-(r+\rho)\Delta} - e^{-2\Delta r}}{r-\rho}\right]\end{aligned}$$

and

$$\begin{aligned}y\int_{t^*}^{t^*+2\Delta} e^{-rt}dt &= -\frac{y}{r}\left[e^{-r(t^*+2\Delta)} - e^{-rt^*}\right] \\ &= e^{-rt^*}y\left[\frac{1-e^{-r2\Delta}}{r}\right].\end{aligned}$$

This means

$$\int_0^\infty (y_t - y)e^{-rt}dt = -e^{-rt^*}y\Delta_y \tag{4.17}$$

where

$$\Delta_y = \frac{1-e^{-r2\Delta}}{r} - \frac{1-e^{-(r+\rho)\Delta}}{r+\rho} - \frac{e^{-(r+\rho)\Delta} - e^{-2\Delta r}}{r-\rho}.$$

Notice that Δ_y does not depend on the parameters of monetary policy. It measures the percentage of the present value (at time t^*) of output that is lost as a result of the post-attack recession. Thus, the fourth term is given by $\int_0^\infty \tau(y_t - y)e^{-rt}dt = -\tau e^{-rt^*}y\Delta_y$.

Fifth term.—For the fifth part of (4.16) we have

$$\begin{aligned}\int_0^\infty \left(\frac{P_t^{NT}g^{NT}}{S_t} - \frac{g^{NT}}{S}\right)e^{-rt}dt &= g^{NT}\left(\int_0^\infty \frac{P_t^{NT}}{S_t}e^{-rt}dt - \frac{1}{rS}\right) \\ &= g^{NT}\left(\int_0^{t^*} \frac{1}{S}e^{-rt}dt + \int_{t^*}^{T_1} \frac{1}{S_t}e^{-rt}dt + \int_{T_1}^\infty \frac{P_t^{NT}}{S_t}e^{-rt}dt - \frac{1}{rS}\right).\end{aligned}$$

For $t \geq T_1$, recall that P_t^{NT} grows at the same rate as P_t. This implies that $P_t^{NT}/S_t = 1/S_{T_1}$ for $t \geq T_1$. Hence we have

$$\int_0^\infty \left(\frac{P_t^{NT}g^{NT}}{S_t} - \frac{g^{NT}}{S}\right)e^{-rt}dt = g^{NT}\left(-\frac{1}{rS}e^{-rt^*} + \int_{t^*}^{T_1} \frac{1}{S_t}e^{-rt}dt + \frac{1}{rS_{T_1}}e^{-rT_1}\right).$$

We evaluate the remainder of this expression numerically using the solutions for the path of the exchange rate described above.

Sixth term.—The sixth part of (4.16) is

$$\begin{aligned}\int_0^\infty \left(\frac{Z_t}{S_t} - \frac{Z}{S}\right)e^{-rt}dt &= \int_0^\infty \frac{Z_t}{S_t}e^{-rt}dt - \frac{Z}{Sr} \\ &= \int_0^{t^*} \frac{Z}{S}e^{-rt}dt + \int_{t^*}^{T_2} \frac{Z}{S_t}e^{-rt}dt + \int_{T_2}^\infty \frac{Z_t}{S_t}e^{-rt}dt - \frac{Z}{Sr} \\ &= -\frac{Z}{rS}e^{-rt^*} + Z\int_{t^*}^{T_2} \frac{1}{S_t}e^{-rt}dt + \frac{Z}{rS_{T_2}}e^{-rT_2}.\end{aligned}$$

In the last line we have used the fact that Z_t grows at the same rate as P_t (and S_t) for $t \geq T_2$. We evaluate the remainder of this expression numerically using the solutions for the path of the exchange rate described above.

Seventh term.—The seventh part of (4.16) is

$$\begin{aligned}\int_0^\infty \left(\frac{rD}{S_t}-\frac{rD}{S}\right)e^{-rt}dt &= rD\left(\int_0^\infty \frac{1}{S_t}e^{-rt}dt-\frac{1}{rS}\right)\\ &= rD\left(\int_0^{t^*}\frac{1}{S}e^{-rt}dt+\int_{t^*}^{T_1}\frac{1}{S_t}e^{-rt}dt+\int_{T_1}^\infty \frac{1}{S_t}e^{-rt}dt-\frac{1}{rS}\right)\\ &= rD\left(-\frac{1}{rS}e^{-rt^*}+\int_{t^*}^{T_1}\frac{1}{S_t}e^{-rt}dt+\frac{1}{S_{T_1}}e^{\mu T_1}\int_{T_1}^\infty e^{-(\mu+r)t}dt\right)\\ &= rD\left(-\frac{1}{rS}e^{-rt^*}+\int_{t^*}^{T_1}\frac{1}{S_t}e^{-rt}dt+\frac{1}{(\mu+r)S_{T_1}}e^{-rT_1}\right)\end{aligned}$$

We evaluate the remainder of this expression numerically using the solutions for the path of the exchange rate described above.

4.3.2. When $t^* = 0$

The analysis in the previous subsection applies up through equation (4.9). Then we use (4.7), this time with $a = 0$ and $c = T$ to write

$$e^{-rT}b_{\underline{T}} - b_{\bar{0}} = \int_0^T x_t e^{-rt}dt,$$

where $b_{\bar{0}} = \lim_{t\downarrow 0} b_t$. Notice that $b_{\bar{0}}$ is the level of debt immediately after the shock and jump in the money supply at time 0. Using (4.8) and (4.9) this implies

$$\begin{aligned}b_{\bar{0}} &= e^{-rT}(b_T+\Delta m_T^T)-\int_0^T x_t e^{-rt}dt\\ &= -\int_0^\infty x_t e^{-rt}dt+e^{-rT}\Delta m_T^T. \qquad (4.18)\end{aligned}$$

There is a jump in real money balances at time 0 which we denote Δm_0^T. This means that

$$b_0 = b_{\bar{0}} + \Delta m_0^T. \qquad (4.19)$$

So finally we have

$$b_0 = -\int_0^\infty x_t e^{-rt}dt + \Delta m_0^T + e^{-rT}\Delta m_T^T. \qquad (4.20)$$

Since $rb_0 = -x$ we can rewrite this as

$$\int_0^\infty (x_t - x)e^{-rt}dt = e^{-rt^*}\Delta m_0^T + e^{-rT}\Delta m_T^T. \qquad (4.21)$$

This means that when $t^* = 0$, we can once again use equation (4.16). As before wemust then simplify (4.16). First, we note that $\int_0^\infty (v_t - v)e^{-rt}dt = \phi$. Then we simplify terms on the right hand side.

For the first term on the right hand side we may again write

$$\int_0^\infty (\dot{m}_t^T + \pi_t^T m_t^T)e^{-rt}dt = \int_0^{T_1} (\dot{m}_t^T + \pi_t^T m_t^T)e^{-rt}dt + \frac{\mu m_{T_1}^T}{r}e^{-rT_1}.$$

Once more, we note that $\dot{m}_t^T + \pi_t^T m_t^T = \dot{M}_t/S_t$. For $0 < t < T$, $M_t = Me^{-\chi}$, the level of money balances right after the speculative attack, so that $\dot{M}_t = 0$. As above, we end up with

$$\int_0^\infty (\dot{m}_t^T + \pi_t^T m_t^T)e^{-rt}dt = \mu M_T e^{-\mu T} \int_T^{T_1} \frac{1}{S_t} e^{(\mu - r)t}dt + \frac{\mu m_{T_1}^T}{r}e^{-rT_1}.$$

For the fourth, fifth, sixth and seventh terms we get identical expressions as before, while noting that $t^* = 0$.

4.4. Summary of Formulas that Must be Computed Numerically

In the first part of the formula we need (1) $\int_T^{T_1} S_t^{-1}e^{(\mu-r)t}dt$ and (2) S_{T_1} since $m_{T_1}^T = M_{T_1}^T/S_{T_1} = Me^{\gamma+\mu(T_1-T)}/S_{T_1}$. In the second part of the formula we need $\Delta m_{t^*}^T$ which is $(Me^{-\chi} - M)/S$ for the case where $t^* > 0$ but is $(Me^{-\chi}/S_{\bar{0}} - M/S)$, where $S_{\bar{0}} = \lim_{t \downarrow 0} S_t$, for the case where $t^* = 0$. Hence we need (3) $S_{\bar{0}}$ in that case. In the third part for the formula we need Δm_T^T which is $(Me^\gamma - Me^{-\chi})/S_T$ which means we need an expression for (4) S_T. In the fifth part of the formula we need the expression for (5) $\int_{t^*}^{T_1} S_t^{-1}e^{-rt}dt$. In the sixth part of the formula we need (6) $\int_{t^*}^{T_2} S_t^{-1}e^{-rt}dt$ and (7) S_{T_2}. To evaluate these expressions we must solve for the path of prices.

4.5. Determining Prices

4.5.1. Determining the CPI

Since, by assumption, $P_t^{NT} = 1$ for $t \leq T_1$ and $T_1 \geq T \geq t^*$, we have $P_t = (S + \delta)^\omega$ for $t < t^*$. Since the exchange rate must follow a continuous path (when $t^* > 0$) we also have $S_{t^*} = S$ and $P_{t^*} = (S + \delta)^\omega$. Of course, if $t^* = 0$, we need to solve for the price level as below.

As we saw above, once the currency is floated, and for all $t \geq t^*$, the price level is determined by (2.1), which we rewrite here:

$$\ln P_t = \eta r - \ln \theta + \frac{1}{\eta} \int_t^\infty e^{-(s-t)/\eta} \ln(M_s/y_s)ds.$$

We consider three cases each of which has subcases.

Case 1.— $t \geq t^* + 2\Delta$. In this case this (2.1) can be written as

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln(M_s/y)ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \frac{\ln y}{\eta}\int_t^\infty e^{-(s-t)/\eta}ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \ln y.
\end{aligned}$$

a.) $t \geq T$. With this additional assumption we have $M_s = M_T e^{\mu(s-T)}$ for $s \geq t$. Hence,

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta - \ln y + \frac{\ln M_T - \mu T}{\eta}\int_t^\infty e^{-(s-t)/\eta}ds + \frac{\mu}{\eta}\int_t^\infty s e^{-(s-t)/\eta}ds \\
&= \eta r - \ln\theta - \ln y + \ln M + \gamma + \mu\eta + \mu(t-T) \\
&= \ln P + \gamma + \mu\eta + \mu(t-T)
\end{aligned}$$

b.) $t^* + 2\Delta \leq t < T$. With this additional assumption we have $M_s = Me^{-\chi}$. Hence,

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta - \ln y + \frac{\ln M - \chi}{\eta}\int_t^T e^{-(s-t)/\eta}ds + \\
&\qquad \frac{\ln M_T - \mu T}{\eta}\int_T^\infty e^{-(s-t)/\eta}ds + \frac{\mu}{\eta}\int_T^\infty s e^{-(s-t)/\eta}ds \\
&= \eta r - \ln\theta - \ln y + (\ln M - \chi)[1 - e^{(t-T)/\eta}] + (\ln M + \gamma + \mu\eta)e^{(t-T)/\eta} \\
&= \ln P - \chi + (\gamma + \chi + \mu\eta)e^{(t-T)/\eta}.
\end{aligned}$$

Case 2.— $t^* + \Delta \leq t < t^* + 2\Delta$. In this case this (2.1) can be written as

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \\
&\qquad \frac{1}{\eta}\int_t^{t^*+2\Delta} e^{-(s-t)/\eta}\ln y_s ds - \frac{1}{\eta}\int_{t^*+2\Delta}^\infty e^{-(s-t)/\eta}\ln y ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \\
&\qquad \frac{1}{\eta}\int_t^{t^*+2\Delta} e^{-(s-t)/\eta}\ln\left\{y e^{\rho[s-(t^*+2\Delta)]}\right\}ds - \frac{1}{\eta}\int_{t^*+2\Delta}^\infty e^{-(s-t)/\eta}\ln y ds
\end{aligned}$$

where we have substituted in the definition of y_s for $t^* + \Delta \leq s \leq t^* + 2\Delta$. Our last equation can be rewritten as

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln y ds - \\
&\quad \frac{\rho}{\eta}\int_t^{t^*+2\Delta}[s-(t^*+2\Delta)]e^{-(s-t)/\eta}ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \frac{\rho(t^*+2\Delta)}{\eta}\int_t^{t^*+2\Delta} e^{-(s-t)/\eta}ds - \frac{\rho}{\eta}\int_t^{t^*+2\Delta} se^{-(s-t)/\eta}ds.
\end{aligned}$$

Finally, using (4.3) and (4.4) we have

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \rho(t^*+2\Delta)(1-e^{[t-(t^*+2\Delta)]/\eta}) + \rho[(t^*+2\Delta+\eta)\, e^{[t-(t^*+2\Delta)]/\eta} - (t+\eta)]
\end{aligned}$$

or

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^\infty e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \rho\eta\{e^{[t-(t^*+2\Delta)]/\eta} - 1\} - \rho[t-(t^*+2\Delta)].
\end{aligned}$$

Since the first four parts of this formula are the same as in Case 1, we immediately get the two subcases.

a.) $t \geq T$. With this additional assumption we have

$$\begin{aligned}
\ln P_t &= \ln P + \gamma + \mu\eta + \mu(t-T) + \\
&\quad \rho\eta\{e^{[t-(t^*+2\Delta)]/\eta} - 1\} - \rho[t-(t^*+2\Delta)].
\end{aligned}$$

b.) $t^* + \Delta \leq t < T$. With this additional assumption we have

$$\begin{aligned}
\ln P_t &= \ln P - \chi + (\gamma+\chi+\mu\eta)e^{(t-T)/\eta} + \\
&\quad \rho\eta\{e^{[t-(t^*+2\Delta)]/\eta} - 1\} - \rho[t-(t^*+2\Delta)].
\end{aligned}$$

Case 3.— $t^* \leq t < t^* + \Delta$. In this case this (2.1) can be written as

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \frac{1}{\eta}\int_t^{t^*+\Delta} e^{-(s-t)/\eta}\ln y_s ds \\
&\quad \frac{1}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(s-t)/\eta}\ln y_s ds - \frac{1}{\eta}\int_{t^*+2\Delta}^{\infty} e^{-(s-t)/\eta}\ln y ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \frac{1}{\eta}\int_t^{t^*+\Delta} e^{-(s-t)/\eta}\ln\left\{ye^{-\rho(s-t^*)}\right\} ds \\
&\quad \frac{1}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(s-t)/\eta}\ln\left\{ye^{\rho[s-(t^*+2\Delta)]}\right\} ds - \frac{1}{\eta}\int_{t^*+2\Delta}^{\infty} e^{-(s-t)/\eta}\ln y ds
\end{aligned}$$

where we have substituted in the definitions of y_s for $t^* \leq s < t^*+\Delta$ and $t^*+\Delta \leq s \leq t^*+2\Delta$. Our last equation can be rewritten as

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln y ds + \\
&\quad \frac{\rho}{\eta}\int_t^{t^*+\Delta} (s-t^*)e^{-(s-t)/\eta} ds - \frac{\rho}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} [s-(t^*+2\Delta)]e^{-(s-t)/\eta} ds \\
&= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \frac{\rho(t^*+2\Delta)}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} e^{-(s-t)/\eta} ds - \frac{\rho t^*}{\eta}\int_t^{t^*+\Delta} e^{-(s-t)/\eta} ds + \\
&\quad \frac{\rho}{\eta}\int_t^{t^*+\Delta} s e^{-(s-t)/\eta} ds - \frac{\rho}{\eta}\int_{t^*+\Delta}^{t^*+2\Delta} s e^{-(s-t)/\eta} ds
\end{aligned}$$

Finally, using (4.3) and (4.4) we have

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \rho(t^*+2\Delta)\left\{e^{[t-(t^*+\Delta)]/\eta} - e^{[t-(t^*+2\Delta)]/\eta}\right\} + \rho t^*(e^{[t-(t^*+\Delta)]/\eta} - 1) + \\
&\quad \rho\left\{(t+\eta) - (t^*+\Delta+\eta)\, e^{[t-(t^*+\Delta)]/\eta}\right\} + \\
&\quad \rho\left\{(t^*+2\Delta+\eta)\, e^{[t-(t^*+2\Delta)]/\eta} - (t^*+\Delta+\eta)\, e^{[t-(t^*+\Delta)]/\eta}\right\}.
\end{aligned}$$

or

$$\begin{aligned}
\ln P_t &= \eta r - \ln\theta + \frac{1}{\eta}\int_t^{\infty} e^{-(s-t)/\eta}\ln M_s ds - \ln y + \\
&\quad \rho\eta\left\{1 - 2e^{[t-(t^*+\Delta)]/\eta} + e^{[t-(t^*+2\Delta)]/\eta}\right\} + \rho(t-t^*).
\end{aligned}$$

Since the first four parts of this formula are the same as in Case 1, we immediately get the two subcases.

a.) $t \geq T$. With this additional assumption we have

$$\begin{aligned}\ln P_t &= \ln P + \gamma + \mu\eta + \mu(t-T) + \\ &\quad \rho\eta\left\{1 - 2e^{[t-(t^*+\Delta)]/\eta} + e^{[t-(t^*+2\Delta)]/\eta}\right\} + \rho(t-t^*)\end{aligned}$$

b.) $t^* \leq t < T$. With this additional assumption we have

$$\begin{aligned}\ln P_t &= \ln P - \chi + (\gamma + \chi + \mu\eta)e^{(t-T)/\eta} + \\ &\quad \rho\eta\left\{1 - 2e^{[t-(t^*+\Delta)]/\eta} + e^{[t-(t^*+2\Delta)]/\eta}\right\} + \rho(t-t^*).\end{aligned}$$

4.5.2. The Price of Nontradables

For $t \leq T_1$, $P_t^{NT} = 1$. For $t > T_1$, we have P_t^{NT} growing at the same rate as P_t. Hence, for $t > T_1$, $P_t^{NT}/P_t = P_{T_1}^{NT}/P_{T_1} = 1/P_{T_1}$, implying that $P_t^{NT} = P_t/P_{T_1}$.

4.5.3. The Exchange Rate

The exchange rate always satisfies $P_t = (S_t + \delta P_t^{NT})^{\omega}(P_t^{NT})^{1-\omega}$, implying that

$$S_t = \left\{P_t^{1/\omega}(P_t^{NT})^{1-1/\omega}\right\} - \delta P_t^{NT}.$$

4.6. Initial Lower and Upper Bounds for μ

Our algorithm uses a simple bracketing method for finding the solution for μ given T and γ. The initial lower bound, $\underline{\mu}$, is 0. The initial upper bound is set equal to a value of μ that would likely provide more seignorage than necessary to finance the increased deficit. If we ignored the effects of the recession, price stickiness in tradables, nominal transfers and debt, and the jumps in the money supply that take place at times t^* and T, and assumed that $M_T = M$ rather than Me^{γ}, (2.9) would be

$$\phi = \mu m_T^T \int_T^{\infty} e^{-rt}dt = \mu m_T^T \frac{1}{r}e^{-rT}.$$

If we also assumed $m_T^T = M/S$ we would have an equation that could be solved for μ:

$$\tilde{\mu} = \phi\frac{S}{M}re^{rT}.$$

Since this is not necessarily an upper bound for the equilibrium value of μ we set our initial guess equal to 100 times this value, i.e. we set $\bar{\mu} = 10\tilde{\mu}$. We set our initial guess equal to $\tilde{\mu}$.