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Solution of Infinite-Horizon Multivariate Linear Rational Expectations Models and Sparse Linear Systems

In this note, we illustrate how the two numerical schemes discussed in our paper "Solution of Finite-Horizon Multivariate Linear Rational Expectations Models and Sparse Linear Systems" may be used to solve infinite-horizon multivariate linear rational expectations (MLRE) models. The note shows how to apply the numerical schemes of Proposition 5.1 (based on the **LDU**-factorization) and of Proposition 5.2 (based on Bowden's procedure) to the solution of a simple infinite-horizon stochastic growth model. This example is meant to be pedagogical. We have chosen it for this note because it shows that on occasions there may be a choice between solving the MLRE model under consideration, if it has a singular coefficient matrix $\mathbf{B}$, by direct application of Proposition 5.1, or by first transforming it so that the coefficient matrix $\mathbf{B}$ is rendered nonsingular and Proposition 5.2 can be applied.

Proposition 5.1 and Proposition 5.2 can be applied to infinite-horizon MLRE models by noting that if the MLRE model

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}E\left(\mathbf{x}_{t+1}|\Omega_t\right) + \mathbf{w}_t, \tag{1}$$

has a unique stable solution, then the solution $\mathbf{x}_t$ does not depend on $E\left(\mathbf{x}_{T+1}|\Omega_t\right)$ for a large enough value of $T-t$. In practice, one will have to solve for $\mathbf{x}_t$ using different values of $T-t$ and $E\left(\mathbf{x}_{T+1}|\Omega_t\right)$ (proceeding recursively forward), and inspect the sensitivity of the solution to these choices.

Consider the following social planning problem describing a standard stochastic growth model:

$$\max_{\{c_0, c_1, \ldots\}} E\left\{\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\eta}}{1-\eta}\right)|\Omega_0\right\} \tag{2}$$

subject to

$$c_t + k_{t+1} - (1-\delta)\,k_t = A_t k_t^{1-\alpha} N_t^{\alpha}, \qquad 0<\alpha<1, \quad 0<\delta<1, \tag{3}$$

$$\log A_t = \rho_0 + \rho_1 \log A_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} N\left(0,\, \sigma^2\right), \tag{4}$$

with the initial conditions k_0 and A_{-1} as given. Here, c_t denotes consumption at time t, k_t the capital stock at the start of the period t, N_t represents labor input, A_t the index of technological progress, β the discount factor, and the information set Ω_t is specified to contain:

$$\Omega_t = \left\{c_t,\, c_{t-1},\, \ldots;\, k_{t+1},\, k_t,\, \ldots;\, N_t,\, N_{t-1},\, \ldots\right\}.$$

We assume that $\{\log A_t\}$ follows a covariance-stationary process ($|\rho_1| < 1$), and normalize the time-endowment so that $N_t = 1$, for all t. Log-linearizing the first-order conditions for this optimization problem around the non-stochastic steady-state values,[1] one obtains the MLRE model

$$\mathbf{x}_t = \mathbf{M}_{00}^{-1}\mathbf{M}_{10}\mathbf{x}_{t-1} + \mathbf{M}_{00}^{-1}\mathbf{M}_{01}E\left(\mathbf{x}_{t+1}|\Omega_t\right) + \mathbf{M}_{00}^{-1}\mathbf{w}_t^*, \tag{5}$$

where

$$\mathbf{x}_t = \begin{pmatrix} \widetilde{c}_t \\ \widetilde{k}_{t+1} \end{pmatrix}, \quad \mathbf{w}_t^* = \begin{pmatrix} \widetilde{A}_t \\ -\left(\beta\left(1-\alpha\right)\overline{Ak}^{-\alpha}\right)E\left(\widetilde{A}_{t+1}|\Omega_t\right) \end{pmatrix},$$

$$\mathbf{M}_{00} = \begin{pmatrix} s_c & s_k \\ \eta & -\alpha\beta\left(1-\alpha\right)\overline{Ak}^{-\alpha} \end{pmatrix}, \quad \mathbf{M}_{10} = \begin{pmatrix} 0 & \left(1-\delta\right)s_k + \left(1-\alpha\right) \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{M}_{01} = \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix},$$

with $s_c = \overline{c}/\left(\overline{Ak}^{1-\alpha}\right)$, $s_k = \overline{k}/\left(\overline{Ak}^{1-\alpha}\right)$, $\widetilde{c}_t = \log\left(c_t/\overline{c}\right)$, $\widetilde{k}_t = \log\left(k_t/\overline{k}\right)$, $\widetilde{A}_t = \log\left(A_t/\overline{A}\right)$, and the non-stochastic steady-state values $\overline{A}$, $\overline{k}$, and $\overline{c}$ satisfy the following equations:

$$\overline{A} = \exp\left(\frac{\rho_0}{1-\rho_1} + \frac{1}{2}\frac{\sigma^2}{1-\rho_1^2}\right), \quad \overline{k} = \left(\frac{\beta\left(1-\alpha\right)\overline{A}}{1-\beta\left(1-\delta\right)}\right)^{1/\alpha},$$

and

$$\overline{c} = \overline{Ak}^{1-\alpha} - \delta\overline{k}.$$

Equation system (5) is a special case of the MLRE model (1) with $\mathbf{A} = \mathbf{M}_{00}^{-1}\mathbf{M}_{10}$, $\mathbf{B} = \mathbf{M}_{00}^{-1}\mathbf{M}_{01}$, and $\mathbf{w}_t = \mathbf{M}_{00}^{-1}\mathbf{w}_t^*$. From the definition of $\mathbf{M}_{01}$, it is clear that the matrix $\mathbf{B}$ for the stochastic growth model considered here is singular. To solve the MLRE model (5) without further transformations, one needs to apply Proposition 5.1, assuming there is a unique stable solution.

Alternatively one could eliminate $\widetilde{c}_t$ from (5), to obtain the MLRE model

$$k_{t+1} = \widehat{\mathbf{M}}_{00}^{-1}\widehat{\mathbf{M}}_{10}\widetilde{k}_t + \widehat{\mathbf{M}}_{00}^{-1}\widehat{\mathbf{M}}_{01}E\left(\widetilde{k}_{t+2}|\Omega_t\right) + \widehat{\mathbf{w}}_t, \tag{6}$$

where

$$\widehat{\mathbf{M}}_{00} = -\eta k/c - \alpha\beta\left(1-\alpha\right)\overline{Ak}^{-\alpha} - \eta\left(\left(1-\delta\right)s_k + \left(1-\alpha\right)\right)/s_c,$$

$$\widehat{\mathbf{M}}_{10} = -\eta\left(\left(1-\delta\right)s_k + \left(1-\alpha\right)\right)/s_c, \quad \widehat{\mathbf{M}}_{01} = -\eta k/c,$$

and

$$\widehat{\mathbf{w}}_t = -\left(\eta/s_c\right)\widetilde{A}_t + \left(\eta/s_c - \beta\left(1-\alpha\right)\overline{Ak}^{-\alpha}\right)E\left(\widetilde{A}_{t+1}|\Omega_t\right).$$

[1] Given the log-linear decision rule setting, specifying the innovations to technology, ε_t, as iid normal does not cause problems for the existence of a time-invariant steady-state probability distribution function for consumption and the capital stock.

Equation (6) is now a special case of the MLRE model (1) with $\mathbf{A} = \widehat{\mathbf{M}}_{00}^{-1}\widehat{\mathbf{M}}_{10}$, $\mathbf{B} = \widehat{\mathbf{M}}_{00}^{-1}\widehat{\mathbf{M}}_{01}$, and $\mathbf{w}_t = \widehat{\mathbf{M}}_{00}^{-1}\widehat{\mathbf{w}}_t$, and has a scalar coefficient matrix $\mathbf{B}$, which is (trivially) nonsingular. Assuming (6) has a unique stable solution, this solution can be computed by applying Proposition 5.2.

For our numerical illustration, we choose the following parameter values, which ensure that (5) and (6) have unique stable solutions: $\beta = (1.03)^{-1/4}$, $\eta = .8$, $\alpha = .64$, $\delta = .02$, $\rho_0 = .15$, $\rho_1 = .96$, and $\sigma = .02$. Using these parameter values, we obtain the following decision rules upon solving (5) using Proposition 5.1, and solving (6) using Proposition 5.2 (and using in this latter case the resultant decision rule for $\widetilde{k}_t$ to obtain the decision rule for $\widetilde{c}_t$):

$$\begin{pmatrix} \widetilde{c}_t \\ \widetilde{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & .6825 \\ 0 & .9691 \end{pmatrix} \begin{pmatrix} \widetilde{c}_{t-1} \\ \widetilde{k}_t \end{pmatrix} + \begin{pmatrix} .2637 \\ .0613 \end{pmatrix} \widetilde{A}_t. \tag{7}$$

In this simple example, the CPU-times required with our programs for obtaining (7) using Proposition 5.1 or Proposition 5.2 differ only insignificantly. It is clear, however, that for higher-dimensional problems the numerical scheme of Proposition 5.2 may be more efficient than that of Proposition 5.1, if the MLRE model under consideration may be transformed into a model with a nonsingular coefficient matrix $\mathbf{B}$. Either procedure only involves elementary matrix operations (addition, multiplication, inversion) but no matrix similarity transformations, and thus can be an attractive alternative to similarity transformation based non-recursive procedures.